Octavian Cira   Florentin Smarandache

VARIOUS ARITHMETIC FUNCTIONS

AND THEIR APPLICATIONS

$$k := 1..last(N)\,,$$

$$N_k\,factor \rightarrow \begin{pmatrix} 3 \\ 3\cdot 19 \\ 111317 \\ 3\cdot 6410977 \\ 7\cdot 577\cdot 926327 \\ 13\cdot 45859055183 \\ 3251\cdot 2580735469940359 \\ 3\cdot 41\cdot 467\cdot 1969449193731640277 \\ 7\cdot 1931\cdot 47123\cdot 2095837\cdot 122225561597 \end{pmatrix}.$$

Octavian Cira and Florentin Smarandache

Various Arithmetic Functions and their Applications

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Preface

Over 300 sequences and many unsolved problems and conjectures related to them are presented herein. These notions, definitions, unsolved problems, questions, theorems corollaries, formulae, conjectures, examples, mathematical criteria, etc. on integer sequences, numbers, quotients, residues, exponents, sieves, pseudo-primes squares cubes factorials, almost primes, mobile periodicals, functions, tables, prime square factorial bases, generalized factorials, generalized palindromes, so on, have been extracted from the Archives of American Mathematics (University of Texas at Austin) and Arizona State University (Tempe): "*The Florentin Smarandache papers*" special collections, University of Craiova Library, and Arhivele Statului (Filiala Vâlcea & Filiala Dolj, România).

The book is based on various articles in the theory of numbers (starting from 1975), updated many times. Special thanks to C. Dumitrescu and V. Seleacufrom the University of Craiova (see their edited book "*Some Notions and Questions in Number Theory*", Erhus Press, Glendale, 1994), M. Bencze, L. Tutescu, E. Burton, M. Coman, F. Russo, H. Ibstedt, C. Ashbacher, S. M. Ruiz, J. Sandor, G. Policarp, V. Iovan, N. Ivaschescu, etc. who helped incollecting and editing this material.

This book was born from the collaboration of the two authors, which started in 2013. The first common work was the volume "*Solving Diophantine Equations*", published in 2014. The contribution of the authors can be summarized as follows: Florentin Smarandache came with his extraordinary ability to propose new areas of study in number theory, and Octavian Cira – with his algorithmic thinking and knowledge of Mathcad.

The work has been edited in LaTeX.

March 23, 2016

Authors

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Introduction

In this we will analyze other functions than the classical functions, [Hardy and Wright, 2008]:

- Multiplicative and additive functions;
- Ω, ω, ν_p – prime power decomposition;
- Multiplicative functions:
  - σ_k, τ, d – divisor sums,
  - φ – Euler totient function,
  - J_k – Jordan totient function,
  - μ – Möbius function,
  - τ – Ramanujan τ function,
  - c_q – Ramanujan's sum;
- Completely multiplicative functions:
  - λ – Liouville function,
  - χ – characters;
- Additive functions:
  - ω – distinct prime divisors;
- Completely additive functions:
  - Ω – prime divisors,
  - ν_p – prime power dividing n;
- Neither multiplicative nor additive:
  - π, Π, θ, ψ – prime count functions,

  - Λ – von Mangoldt function,
  - p – partition function,
  - λ – Carmichael function,
  - h – Class number,
  - r_k – Sum of k squares;

- Summation functions,
- Dirichlet convolution,
- Relations among the functions;
  - Dirichlet convolutions,
  - Sums of squares,
  - Divisor sum convolutions,
  - Class number related,
  - Prime–count related,
  - Menon's identity.

In the book we have extended the following functions:

- of counting the digits in base b,
- digits the number in base b,
- primes counting using Smarandache's function,
- multifactorial,
- digital
  - sum in base b,
  - sum in base b to the power k,
  - product in base b,
- divisor product,
- proper divisor product,
- n–multiple power free sieve,
- irrational root sieve,

- n–ary power sieve,
- k–ary consecutive sieve,
- consecutive sieve,
- prime part, square part, cubic part, factorial part, function part,
- primorial,
- Smarandache type functions:
  - Smarandache–Cira function of order k,
  - Smarandache–Kurepa,
  - Smarandache–Wagstaff,
  - Smarandache near to k–primorial,
  - Smarandache ceil,
  - Smarandache–Mersenne,
  - Smarandache–X-nacci,
  - pseudo–Smarandache,
  - alternative pseudo–Smarandache,
  - Smarandache functions of the k-th kind,
- factorial for real numbers,
- analogues of the Smarandache,
- $m-$powers,
- and we have also introduced alternatives of them.

The next chapter of the book is dedicated to primes. Algorithms are presented: sieve of Eratosthenes, sieve of Sundram, sieve of Atkin. In the section dedicated to the criteria of primality, the Smarandache primality criterion is introduced. The next section concentrates on Luhn prime numbers of first, second and third rank. The odd primes have the final digits 1, 3, 7 or 9. Another section studies the number of primes' final digits. The difference between two primes is called gap. It seems that gaps of length 6 are the most numerous. In the last section, we present the polynomial which generates primes.

The second chapter (the main chapter of the this book) is dedicated to arithmetical functions.

The third chapter is dedicated to numbers' sequences: consecutive sequence, circular sequence, symmetric sequence, deconstructive sequence, concatenated sequences, permutation sequence, combinatorial sequences.

The fourth chapter discusses special numbers. The first section presents numeration bases and Smarandache numbers, Smarandache quotients, primitive numbers, $m-$power residues, exponents of power m, almost prime, pseudo–primes, permutation–primes, pseudo–squares, pseudo–cubes, pseudo–m–powers, pseudo–factorials, pseudo–divisors, pseudo–odd numbers, pseudo–triangular numbers, pseudo–even numbers, pseudo–multiples of $prime$, progressions, palindromes, Smarandache–Wellin primes.

The fifth chapter treats about a series of numbers that have applicability in sciences.

The sixth chapter approximates some constants that are connected to the series proposed in this volume.

The carpet numbers are discussed in the seventh chapter, suggesting some algorithms to generate these numbers.

All open issues that have no confirmation were included in the eighth chapter, *Conjecture*.

The ninth chapter includes algorithms that generate series of numbers with some special properties.

The tenth chapter comprises some Mathcad documents that have been created for this volume. For the reader interested in a particular issue, we provide an adequate Mathcad document.

The book includes a chapter of *Indexes*: notation, Mathcad utility functions used in this work, user functions that have been called in this volume, series generation programs and an index of names.

Chapter 1

Prime Numbers

1.1 Generating Primes

Generating primes can be obtained by means of several deterministic algorithms, known in the literature as sieves: Eratosthenes, Euler, Sundaram, Atkin, etc.

1.1.1 Sieve of Eratosthenes

The linear variant of the Sieve of Eratosthenes implemented by Pritchard [1987], given by the code, has the inconvenience that it uselessly repeats operations.

Our implementation optimizes Pritchard's algorithm, lowering to minimum the number of putting to zero in the vector *is_prime* and reducing to maximum the used memory. The speed of generating primes up to the limit L is remarkable.

Program 1.1. $SEPC$ (Sieve of Erathostenes, linear version of Prithcard, optimized of Cira) program of generating primes up to L.

$$
\begin{aligned}
SEPC(L) :=\ & \lambda \leftarrow floor\left(\frac{L}{2}\right)\\
& for\ j \in 1..\lambda\\
& \quad is\_prime_j \leftarrow 1\\
& prime \leftarrow (2\ 3\ 5\ 7)^{\mathrm{T}}\\
& i \leftarrow 5\\
& for\ j \in 4, 7..\lambda\\
& \quad is\_prime_j \leftarrow 0\\
& k \leftarrow 3\\
& while\ (prime_k)^2 \le L
\end{aligned}
$$

$$
\begin{array}{|l}
\quad \begin{array}{|l}
f \leftarrow \dfrac{(prime_k)^2-1}{2} \\
for\ j \in f, f+prime_k..\lambda \\
\quad is\_prime_j \leftarrow 0 \\
s \leftarrow \dfrac{(prime_{k-1})^2+1}{2} \\
for\ j \in s..f \\
\quad if\ is\_prime_j=1 \\
\quad\quad \begin{array}{|l} prime_i \leftarrow 2\cdot j+1 \\ i \leftarrow i+1 \end{array} \\
k \leftarrow k+1
\end{array} \\
j \leftarrow f \\
while\ \ j<\lambda \\
\quad \begin{array}{|l}
if\ is\_prime_j=1 \\
\quad \begin{array}{|l} prime_i \leftarrow 2\cdot j+1 \\ i \leftarrow i+1 \end{array} \\
j \leftarrow j+1
\end{array} \\
return\ \ prime
\end{array}
$$

It is known that for $L<10^{10}$ the Erathostene's sieve in the linear variant of Pritchard is the fastest primes' generator algorithm, [Cira and Smarandache, 2014]. Then, the program *SEPC*, 1.1, is more performant.

1.1.2 Sieve of Sundaram

The Sieve of Sundaram is a simple deterministic algorithm for finding the primes up to a given natural number. This algorithm was presented by [Sundaram and Aiyar, 1934]. As it is known, the Sieve of Sundaram uses $O(L\log(L))$ operations in order to find the primes up to L. The algorithm of the Sieve of Sundaram in Mathcad is:

Program 1.2.

$$
SS(L) := \begin{array}{|l}
m \leftarrow floor\left(\dfrac{L}{2}\right) \\
for\ k \in 1..m \\
\quad is\_prime_k \leftarrow 1 \\
for\ k \in 1..m \\
\quad for\ j \in 1..ceil\left(\dfrac{m-k}{2\cdot k+1}\right) \\
\quad\quad is\_prime_{k+j+2\cdot k\cdot j} \leftarrow 0 \\
prime_1 \leftarrow 2 \\
j \leftarrow 1
\end{array}
$$

$$
\begin{array}{|l}
for\ k \in 1..m \\
\quad if\ is\_prime_k{=}1 \\
\qquad \left| \begin{array}{l} j \leftarrow j+1 \\ prime_j \leftarrow 2\cdot k+1 \end{array} \right. \\
return\ prime
\end{array}
$$

1.1.3 Sieve of Atkin

Until recently, i.e. till the appearance of the Sieve of Atkin, [Atkin and Bernstein, 2004], the Sieve of Eratosthenes was considered the most efficient algorithm that generates all the primes up to a limit $L > 10^{10}$.

Program 1.3. *SAOC* (Sieve of Atkin Optimized of Cira) program of generating primes up to L.

$$
SAOC(L) := \left| \begin{array}{l}
is\_prime_L \leftarrow 0 \\
\lambda \leftarrow floor\left(\sqrt{L}\right) \\
for\ j \in 1..ceil(\lambda) \\
\quad \left| \begin{array}{l}
for\ k \in 1..ceil\left(\dfrac{\sqrt{L-j^2}}{2}\right) \\
\quad \left| \begin{array}{l} n \leftarrow 4k^2 + j^2 \\ m \leftarrow \mod(n,12) \\ is\_prime_n \leftarrow \neg is\_prime_n\ \ if\ n \le L \wedge (m{=}1 \vee m{=}5) \end{array} \right. \\
for\ k \in 1..ceil\left(\sqrt{\dfrac{L-j^2}{3}}\right) \\
\quad \left| \begin{array}{l} n \leftarrow 3k^2 + j^2 \\ is\_prime_n \leftarrow \neg is\_prime_n\ \ if\ n \le L \wedge \mod(n,12){=}7 \end{array} \right. \\
for\ k \in j+1..ceil\left(\sqrt{\dfrac{L+j^2}{3}}\right) \\
\quad \left| \begin{array}{l} n \leftarrow 3k^2 - j^2 \\ is\_prime_n \leftarrow \neg is\_prime_n\ \ if\ n \le L \wedge \mod(n,12){=}11 \end{array} \right.
\end{array} \right. \\
for\ j \in 5,7..\lambda \\
\quad for\ k \in 1,3..\dfrac{L}{j^2}\ \ if\ is\_prime_j \\
\qquad is\_prime_{k\cdot j^2} \leftarrow 0 \\
prime_1 \leftarrow 2 \\
prime_2 \leftarrow 3 \\
for\ n \in 5,7..L \\
\quad if\ is\_prime_n \\
\qquad \left| prime_j \leftarrow n \right.
\end{array} \right.
$$

$$\left|\begin{array}{l} \quad\quad \left| j \leftarrow j+1 \right. \\ return\ \ prime \end{array}\right.$$

1.2 Primality Criteria

1.2.1 Smarandache Primality Criterion

Let $S : \mathbb{N} \to \mathbb{N}$ be Smarandache function [Sondow and Weisstein, 2014], [Smarandache, 1999a,b], that gives the smallest value for a given n at which $n \mid S(n)!$ (i.e. n divides $S(n)!$).

Theorem 1.4. *Let n be an integer $n > 4$. Then n is prime if and only if $S(n) = n$.*

Proof. See [Smarandache, 1999b, p. 31]. □

As seen in Theorem 1.4, we can use as primality test the computing of the value of S function. For $n > 4$, if relation $S(n) = n$ is satisfied, it follows that n is prime. In other words, the primes (to which number 4 is added) are fixed points for S function. In this study we will use this primality test.

Program 1.5. The program returns the value 0 if the number is not prime and the value 1 if the number is prime. File $\eta.prn$ (contains the values function Smarandache) is read and assigned to vector η .

$$ORIGIN := 1 \quad \eta := READPRN("\ldots\backslash\eta.prn")$$

$$TS(n) := \left| \begin{array}{l} return\ \text{"Error. } n<1 \text{ or not integer"}\ \ if\ n<1 \vee n \neq trunc(n) \\ if\ n>4 \\ \quad \left| \begin{array}{l} return\ 0\ \ if\ \eta_n \neq n \\ return\ 1\ otherwise \end{array} \right. \\ otherwise \\ \quad \left| \begin{array}{l} return\ 0\ \ if\ n{=}1 \vee n{=}4 \\ return\ 1\ otherwise \end{array} \right. \end{array} \right.$$

By means of the program *TS*, 1.5 was realized the following test.

$$n := 499999 \ \ k := 1..n \ \ v_k := 2 \cdot k + 1$$

$$last(v) = 499999 \ \ v_1 = 3 \ \ v_{last(v)} = 999999$$

$$t_0 := time(0) \ \ w_k := TS(v_k) \ \ t_1 := time(1)$$

$$(t_1 - t_0)sec = 0.304s \quad \sum w = 78497 \ .$$

The number of primes up to 10^6 is 78798, and the sum of non-zero components (equal to 1) is 78797, as 2 was not counted as prime number because it is an even number.

1.3 Luhn primes

The number 229 is the smallest prime which summed with its inverse gives also a prime. Indeed, 1151 is a prime, and $1151 = 229 + 922$. The first to note this special property of 229, on the website *Prime Curios*, was Norman Luhn (9 Feb. 1999), [Luhn, 2013, Caldwell and Honacher Jr., 2014].

Function 1.6. The function that returns the reverse of number $n_{(10)}$ in base b.

$$Reverse(n[,b]) = sign(n) \cdot reverse(dn(|n|,b)) \cdot Vb(b, nrd(|n|,b))\ ,$$

where $reverse(v)$ is the Mathcad function that returns the inverse of vector v, $dn(n,b)$ is the program 2.2 which returns the digits of number $n_{(10)}$ in numeration base b, $nrd(n,b)$ is the function 2.1 which returns the number of digits of the number $n_{(10)}$ in numeration base b, and the program $Vb(b,m)$ returns the vector $(b^m\ b^{m-1}\ \ldots b^0)^{\mathrm{T}}$. If the argument b lacks when calling the function *Reverse* (the notation $[,b]$ shows that the argument is optional) we have a numeration base 10.

Definition 1.7. The primes p for which $p^o + Reverse(p)^o \in \mathbb{P}_{\geq 2}$ are called *Luhn primes* of o rank. We simply call *Luhn primes* of first rank ($o = 1$) simple *Luhn primes*.

Program 1.8. The pL program for determining *Luhn primes* of o rank up to the limit L.

```
pL(o,L) := | return "Error. L<11" if L<11
           | p ← SEPC(L)
           | j ← 1
           | for k ∈ 1..last(p)
           |   | d ← trunc(p_k · 10^(−nrd(p_k,10)+1))
           |   | if d=2 ∨ d=4 ∨ d=6 ∨ d=8
           |   |   | q_j ← p_k
           |   |   | j ← j+1
           | j ← 1
           | for k ∈ 1..last(q)
           |   | s ← (q_k)^o + Reverse(q_k)^o
           |   | if TS(s)=1
           |   |   | v_j ← q_k
           |   |   | j ← j+1
           | return v
```

where TS is the primality Smarandache test, 1.5, and $Reverse(n)$ is the function 1.6. In the first part of the program, the primes that have an odd digit as the first digit are dropped.

1.3.1 Luhn Primes of First Rank

Up to $L < 3 \cdot 10^4$ we have 321 *Luhn primes* of first rank: 229, 239, 241, 257, 269, 271, 277, 281, 439, 443, 463, 467, 479, 499, 613, 641, 653, 661, 673, 677, 683, 691, 811, 823, 839, 863, 881, 20011, 20029, 20047, 20051, 20101, 20161, 20201, 20249, 20269, 20347, 20389, 20399, 20441, 20477, 20479, 20507, 20521, 20611, 20627, 20717, 20759, 20809, 20879, 20887, 20897, 20981, 21001, 21019, 21089, 21157, 21169, 21211, 21377, 21379, 21419, 21467, 21491, 21521, 21529, 21559, 21569, 21577, 21601, 21611, 21617, 21647, 21661, 21701, 21727, 21751, 21767, 21817, 21841, 21851, 21859, 21881, 21961, 21991, 22027, 22031, 22039, 22079, 22091, 22147, 22159, 22171, 22229, 22247, 22291, 22367, 22369, 22397, 22409, 22469, 22481, 22501, 22511, 22549, 22567, 22571, 22637, 22651, 22669, 22699, 22717, 22739, 22741, 22807, 22859, 22871, 22877, 22961, 23017, 23021, 23029, 23081, 23087, 23099, 23131, 23189, 23197, 23279, 23357, 23369, 23417, 23447, 23459, 23497, 23509, 23539, 23549, 23557, 23561, 23627, 23689, 23747, 23761, 23831, 23857, 23879, 23899, 23971, 24007, 24019, 24071, 24077, 24091, 24121, 24151, 24179, 24181, 24229, 24359, 24379, 24407, 24419, 24439, 24481, 24499, 24517, 24547, 24551, 24631, 24799, 24821, 24847, 24851, 24889, 24979, 24989, 25031, 25057, 25097, 25111, 25117, 25121, 25169, 25171, 25189, 25219, 25261, 25339, 25349, 25367, 25409, 25439, 25469, 25471, 25537, 25541, 25621, 25639, 25741, 25799, 25801, 25819, 25841, 25847, 25931, 25939, 25951, 25969, 26021, 26107, 26111, 26119, 26161, 26189, 26209, 26249, 26251, 26339, 26357, 26417, 26459, 26479, 26489, 26591, 26627, 26681, 26701, 26717, 26731, 26801, 26849, 26921, 26959, 26981, 27011, 27059, 27061, 27077, 27109, 27179, 27239, 27241, 27271, 27277, 27281, 27329, 27407, 27409, 27431, 27449, 27457, 27479, 27481, 27509, 27581, 27617, 27691, 27779, 27791, 27809, 27817, 27827, 27901, 27919, 28001, 28019, 28027, 28031, 28051, 28111, 28229, 28307, 28309, 28319, 28409, 28439, 28447, 28571, 28597, 28607, 28661, 28697, 28711, 28751, 28759, 28807, 28817, 28879, 28901, 28909, 28921, 28949, 28961, 28979, 29009, 29017, 29021, 29027, 29101, 29129, 29131, 29137, 29167, 29191, 29221, 29251, 29327, 29389, 29411, 29429, 29437, 29501, 29587, 29629, 29671, 29741, 29759, 29819, 29867, 29989 .

The number of *Luhn primes* up to the limit L is given in Table 1.1:

L	$3 \cdot 10^2$	$5 \cdot 10^2$	$7 \cdot 10^2$	$9 \cdot 10^2$	$3 \cdot 10^4$	$5 \cdot 10^4$	$7 \cdot 10^4$	$9 \cdot 10^4$
	8	14	22	27	321	586	818	1078

Table 1.1: Numbers of Luhn primes

Up to the limit $L = 2 \cdot 10^7$ the number of *Luhn primes* is 50598, [Cira and Smarandache, 2015].

1.3.2 Luhn Primes of Second Rank

Are there *Luhn primes* of second rank? Yes, indeed. 23 is a *Luhn prime* number of second rank because 1553 is a prime and we have $1553 = 23^2 + 32^2$. Up to $3 \cdot 10^4$ we have 158 *Luhn primes* of second rank: 23, 41, 227, 233, 283, 401, 409, 419, 421, 461, 491, 499, 823, 827, 857, 877, 2003, 2083, 2267, 2437, 2557, 2593, 2617, 2633, 2677, 2857, 2887, 2957, 4001, 4021, 4051, 4079, 4129, 4211, 4231, 4391, 4409, 4451, 4481, 4519, 4591, 4621, 4639, 4651, 4871, 6091, 6301, 6329, 6379, 6521, 6529, 6551, 6781, 6871, 6911, 8117, 8243, 8273, 8317, 8377, 8543, 8647, 8713, 8807, 8863, 8963, 20023, 20483, 20693, 20753, 20963, 20983, 21107, 21157, 21163, 21383, 21433, 21563, 21587, 21683, 21727, 21757, 21803, 21863, 21937, 21997, 22003, 22027, 22063, 22133, 22147, 22193, 22273, 22367, 22643, 22697, 22717, 22787, 22993, 23057, 23063, 23117, 23227, 23327, 23473, 23557, 23603, 23887, 24317, 24527, 24533, 24547, 24623, 24877, 24907, 25087, 25237, 25243, 25453, 25523, 25693, 25703, 25717, 25943, 26053, 26177, 26183, 26203, 26237, 26357, 26407, 26513, 26633, 26687, 26987, 27043, 27107, 27397, 27583, 27803, 27883, 28027, 28297, 28513, 28607, 28643, 28753, 28807, 29027, 29063, 29243, 29303, 29333, 29387, 29423, 29537, 29717, 29983 .

Proposition 1.9. *The digit of unit for sum* $q^2 + \text{Reverse}(q)^2$ *is 3 or 7 for all numbers* q Luhn primes *of second rank.*

Proof. The square of an even number is an even number, the square of an odd number is an odd number. The sum of an odd number with an even number is an odd number. If q is a *Luhn prime* number of second rank, then its reverse must be necessarily an even number, because $\sigma = q^2 + Reverse(q)^2$ an odd number. The prime q has unit digit 1, 3, 7 or 9 and *Reverse*(q) will obligatory have the digit unit, 2, 4, 6 or 8. Then, q^2 will have the unit digit, respectively 1, 9, 9 or 1, and unit digit of *Reverse*$(q)^2$ will be respectively 4, 6, 6 or 4. Then, we consider all possible combinations of summing units $1+4=5$, $1+6=7$, $9+4=13$, $9+6=15$. Sums ending in 5 does not suit, therefore only endings of sum σ that does suit, as σ can eventually be a prime, are 3 or 7. □

This sentence can be used to increase the speed determination algorithm of *Luhn prime* numbers of second rank, avoiding the primality test or sums $\sigma = q^2 + Reverse(q)^2$ which end in digit 5.

Up to the limit $L = 3 \cdot 10^4$ *Luhn prime* numbers of o rank, $o = 3$ were not found.

Up to $3 \cdot 10^4$ we have 219 *Luhn prime* numbers of o rank, $o = 4$: 23, 43, 47, 211, 233, 239, 263, 419, 431, 487, 491, 601, 683, 821, 857, 2039, 2063, 2089, 2113, 2143, 2203, 2243, 2351, 2357, 2377, 2417, 2539, 2617, 2689, 2699, 2707, 2749, 2819, 2861, 2917, 2963, 4051, 4057, 4127, 4129, 4409, 4441, 4481, 4603, 4679,

4733, 4751, 4951, 4969, 4973, 6053, 6257, 6269, 6271, 6301, 6311, 6353, 6449, 6547, 6551, 6673, 6679, 6691, 6803, 6869, 6871, 6947, 6967, 8081, 8123, 8297, 8429, 8461, 8521, 8543, 8627, 8731, 8741, 8747, 8849, 8923, 8951, 8969, 20129, 20149, 20177, 20183, 20359, 20369, 20593, 20599, 20639, 20717, 20743, 20759, 20903, 20921, 21017, 21019, 21169, 21211, 21341, 21379, 21419, 21503, 21611, 21613, 21661, 21727, 21803, 21821, 21841, 21881, 21893, 21929, 21937, 22031, 22073, 22133, 22171, 22277, 22303, 22343, 22349, 22441, 22549, 22573, 22741, 22817, 22853, 22877, 22921, 23029, 23071, 23227, 23327, 23357, 23399, 23431, 23531, 23767, 23827, 23917, 23977, 24019, 24023, 24113, 24179, 24197,24223, 24251, 24421, 24481, 24527, 24593, 24659, 24683, 24793, 25171, 25261, 25303, 25307, 25321, 25343, 25541, 25643, 25673, 25819, 25873, 25969, 26083, 26153, 26171, 26267, 26297, 26561, 26833, 26839, 26953, 26993, 27103, 27277, 27337, 27427, 27551, 27617, 27749, 27751, 27791, 27823, 27901, 27919, 27953, 28019, 28087, 28211, 28289, 28297, 28409, 28547, 28631, 28663, 28723, 28793, 28813, 28817, 28843, 28909, 28927, 28949, 28979, 29063, 29173, 29251, 29383, 29663, 29833, 29881, 29989 .

The number $23^4 + 32^4 \to 1328417$ is a prime, the number $43^4 + 34^4 \to 4755137$ is a prime, ..., the number $29989^4 + 98992^4 \to 96837367848621546737$ is a prime.

Remark 1.10. Up to $3 \cdot 10^4$, the numbers: 23, 233, 419, 491, 857, 2617, 4051, 4129, 4409, 4481, 6301, 6551, 6871, 8543, 21727, 21803, 21937, 22133, 23227, 23327, 24527, 28297, 29063 are *Luhn prime* numbers of 2nd and 4th rank.

Questions:

1. There are an infinite number of *Luhn primes* of first rank?
2. There are an infinite number of *Luhn primes* of second rank?
3. There are *Luhn primes* of third rank?
4. There are an infinite number of *Luhn primes* of fourth rank?
5. There are Luhn prime numbers of o rank, $o > 4$?

1.4 Endings the Primes

Primes, with the exception of 2 and 5, have their unit digit equal to 1, 3, 7 or 9. It has been counting the first 200,000 primes, i.e. from 2 to $prime_{200000} = 2750159$. The final units 3 and 7 "dominate" the final digits 1 and 9 to primes.

For this observation, we present the counting program for final digits on primes (Number of appearances as the Final Digit):

Program 1.11. Program for counting the final digits on primes.

$$
nfd(k_{max}) := \left|
\begin{array}{l}
return\ \text{"Error. } k_{max} > last(prime)\text{"}\ \ if\ k_{max} > last(prime)\\
nrd \leftarrow (0\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0)\\
for\ k \in 2..k_{max}\\
\quad \left| \begin{array}{l}
fd \leftarrow prime_k - Trunc(prime_k, 10)\\
for\ j = 1..9\\
\quad \left| \begin{array}{l}
nrd_{k,j} \leftarrow nrd_{k-1,j} + 1\ \ if\ j{=}fd\\
nrd_{k,j} \leftarrow nrd_{k-1,j}\ \ otherwise
\end{array}\right.
\end{array}\right.\\
return\ nrd
\end{array}\right.
$$

Calling the program $nrfd$=$nfd(2 \cdot 10^5)$, it provides a matrix of 9 columns and 200,000 lines. The component $nrfd_{k,1}$ give us the number of primes that have digit unit 1 to the limit $prime_k$, the component $nrfd_{k,3}$ provides the number of primes that have the unit digit 3 up to the limit $prime_k$, and so on. Therefore, we have $nrfd_{15,1} = 3$ (11, 31 and 41 have digit unit 1), $nrfd_{15,3} = 4$ (3, 13, 23 and 43 have digit unit 3), $nrfd_{15,7} = 4$ (7, 17, 37 and 47 have digit unit 7) and $nrfd_{15,9} = 2$ (19 and 29 have digit unit 9). We recall that $prime_{15} = 47$.

For presenting the fact that the digits 3 and 7 "*dominates*" the digits 1 and 9, we use the function $umd : \mathbb{N}^* \to \mathbb{N}^*$ (Upper limit of Mean Digits appearances), $umd(L) = \left\lceil \frac{L}{4} \right\rceil$. This function is the superior limit of the mean of digits 1, 3, 7 appearances and 9 as endings of primes (exception for 2 and 5 which occur each only once).

In Figure 1.1 we present the graphs of functions: $nrfd_{k,1} - umd(k)$ (red), $nrfd_{k,3} - umd(k)$ (blue), $nrfd_{k,7} - umd(k)$ (green) and $nrfd_{k,9} - umd(k)$ (black).

1.5 Numbers of Gap between Primes

Definition 1.12. The difference between two successive primes is called *gap*. The *gap* is denoted by g, but to specify which specific *gap*, it is available the notation $g_n = p_{n+1} - p_n$, where $p_n, p_{n+1} \in \mathbb{P}_{\geq 2}$.

Observation 1.13. There are authors that considers the *gap* is given by the formula $g_n = p_{n+1} - p_n - 1$, where $p_n, p_{n+1} \in \mathbb{P}_{\geq 2}$.

In Table 1.2, it was displayed the number of *gaps* of length 2, 4, 6, 8, 10, 12 for primes lower than $3 \cdot 10^7$, $2 \cdot 10^7$, 10^7, 10^6, 10^5, 10^4 and 10^3.

Let us notice the the *gaps* of length 6 are more frequent than the *gaps* of length 2 or 4. Generally, the *gaps* of multiple of 6 length are more frequent than the *gaps* of comparable length.

Given this observation, we can enunciate the following conjecture: the *gap of length 6 is the most common gap*.

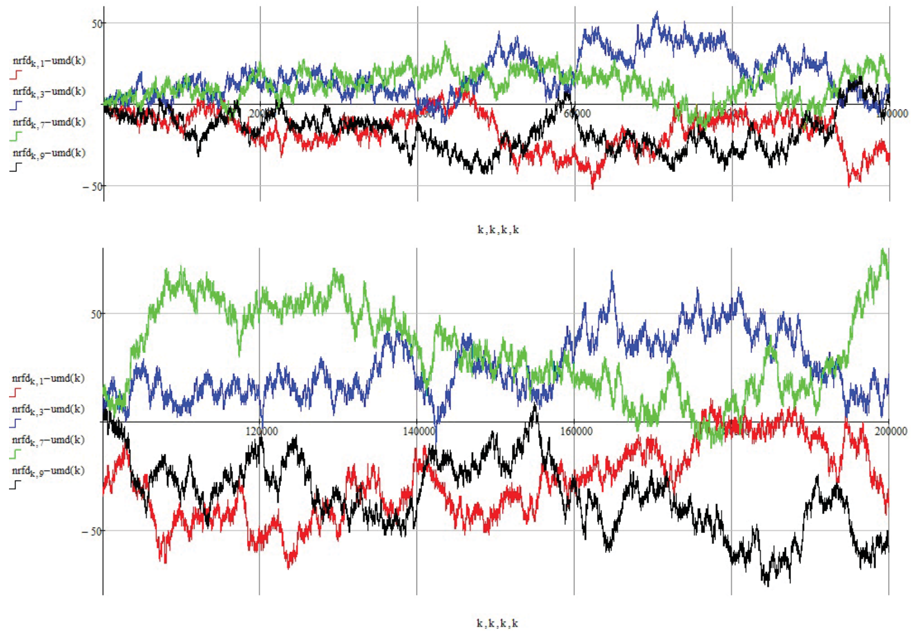

Figure 1.1: Graphic of terminal digits for primes

1.6 Polynomials Generating Prime Numbers

These algebraic polynomials have the property that for $n = 0, 1, \ldots, m-1$ value of the polynomial, eventually in module, are m primes.

1. Polynomial $P(n) = n^3 + n^2 + 17$ generates 11 primes: 17, 19, 29, 53, 97, 167, 269, 409, 593, 827, 1117, [Sloane, 2014, A050266].

2. Polynomial $P(n) = 2n^2 + 11$ generates 11 primes: 11, 13, 19, 29, 43, 61, 83, 109, 139, 173, 211, [Sloane, 2014, A050265].

3. Honaker polynomial, $P(n) = 4n^2 + 4n + 59$, generates 14 primes: 59, 67, 83, 107, 139, 179, 227, 283, 347, 419, 499, 587, 683, 787, [Sloane, 2014, A048988].

4. Legendre polynomial, $P(n) = n^2 + n + 17$, generates 16 primes: 17, 19, 23, 29, 37, 47, 59, 73, 89, 107, 127, 149, 173, 199, 227, 257 , [Wells, 1986], [Sloane, 2014, A007635].

5. Bruno [2009] polynomial, $P(n) = 3n^2 + 39n + 37$, generates 18 primes: 37, 79, 127, 181, 241, 307, 379, 457, 541, 631, 727, 829, 937, 1051, 1171, 1297, 1429, 1567 .

g	$3\cdot 10^7$	$2\cdot 10^7$	10^7	10^6	10^5	10^4	10^3
2	152891	107407	58980	8169	1224	205	35
4	152576	107081	58621	8143	1215	202	40
6	**263423**	**183911**	**99987**	**13549**	**1940**	**299**	**44**
8	113368	78792	42352	5569	773	101	15
10	145525	101198	54431	7079	916	119	16
12	**178927**	**123410**	**65513**	**8005**	**965**	**105**	**8**
14	96571	66762	35394	4233	484	54	7
16	70263	47951	25099	2881	339	33	0
18	**124171**	**84782**	**43851**	**4909**	**514**	**40**	**1**
20	63966	43284	22084	2402	238	15	1

Table 1.2: Number of *gaps* of length 2, 4, ..., 20

6. Pegg Jr. [2005] polynomial, $P(n) = n^4 + 29n^2 + 101$, generates 20 primes: 101, 131, 233, 443, 821, 1451, 2441, 3923, 6053, 9011, 13001, 18251, 25013, 33563, 44201, 57251, 73061, 92003, 114473, 140891 .

7. Gobbo [2005a] polynomial, $P(n) = \left|7n^2 - 371n + 4871\right|$, generates 24 primes: 4871, 4507, 4157, 3821, 3499, 3191, 2897, 2617, 2351, 2099, 1861, 1637, 1427, 1231, 1049, 881, 727, 587, 461, 349, 251, 167, 97, 41 .

8. Legendre polynomial (1798), $P(n) = 2n^2 + 29$ generates 29 primes: 29, 31, 37, 47, 61, 79, 101, 127, 157, 191, 229, 271, 317, 367, 421, 479, 541, 607, 677, 751, 829, 911, 997, 1087, 1181, 1279, 1381, 1487, 1597 , [Sloane, 2014, A007641].

9. Brox [2006] polynomial, $P(n) = 6n^2 - 342n + 4903$, generates 29 primes: 4903, 4567, 4243, 3931, 3631, 3343, 3067, 2803, 2551, 2311, 2083, 1867, 1663, 1471, 1291, 1123, 967, 823, 691, 571, 463, 367, 283, 211, 151, 103, 67, 43, 31 .

10. Gobbo [2005b] polynomial, $P(n) = \left|8n^2 - 488n + 7243\right|$, generates 31 primes: 7243, 6763, 6299, 5851, 5419, 5003, 4603, 4219, 3851, 3499, 3163, 2843, 2539, 2251, 1979, 1723 1483, 1259, 1051, 859, 683, 523, 379, 251, 139, 43, 37, 101, 149, 181, 197 .

11. Brox [2006] polynomial, $P(n) = 43n^2 - 537n + 2971$, generates 35 primes: 2971, 2477, 2069, 1747, 1511, 1361, 1297, 1319, 1427, 1621, 1901, 2267, 2719, 3257, 3881, 4591, 5387, 6269 7237, 8291, 9431, 10657, 11969, 13367, 14851, 16421, 18077, 19819, 21647, 23561, 25561, 27647, 29819, 32077, 34421 .

12. Wroblewski and Meyrignac polynomial, [Wroblewski and Meyrignac, 2006],

$$P(n) = \left|42n^3 + 270n^2 - 26436n + 250703\right| ,$$

generates 40 primes: 250703, 224579, 199247, 174959, 151967, 130523, 110879, 93287, 77999, 65267, 55343, 48479, 44927, 44939 48767, 56663, 68879, 85667, 107279, 133967, 165983, 203579, 247007, 296519, 352367, 414803, 484079, 560447 644159, 735467, 834623, 941879, 1057487, 1181699, 1314767, 1456943, 1608479, 1769627, 1940639, 2121767 .

13. Euler's polynomial (1772), $P(n) = n^2 + n + 41$, generates 40 primes: 41, 43, 47, 53, 61, 71, 83, 97, 113, 131, 151, 173, 197, 223, 251, 281, 313, 347, 383, 421, 461 503, 547, 593, 641, 691, 743, 797, 853, 911, 971, 1033, 1097, 1163, 1231, 1301, 1373, 1447, 1523, 1601, [Sloane, 2014, A005846].

14. Legendre (1798) polynomial, $P(n) = n^2 - n + 41$, generates 40 primes: 41, 41, 43, 47, 53, 61, 71, 83, 97, 113, 131, 151, 173, 197, 223, 251, 281, 313, 347, 383, 421 461, 503, 547, 593, 641, 691, 743, 797, 853, 911, 971, 1033, 1097, 1163, 1231, 1301, 1373, 1447, 1523, 1601 . In the list there are 41 primes, but the number 41 repeats itself.

15. Speiser [2005] polynomial, $P(n) = \left|103n^2 - 4707 + 50383\right|$, generates 43 primes: 50383, 45779, 41381, 37189, 33203, 29423, 25849, 22481, 19319, 16363, 13613, 11069, 8731, 6599 4673, 2953, 1439, 131, 971, 1867, 2557, 3041, 3319, 3391, 3257, 2917, 2371, 1619 661, 503, 1873, 3449, 5231, 7219, 9413, 11813, 14419, 17231, 20249, 23473, 26903, 30539, 34381 .

16. Fung and Ruby polynomial, [Fung and Williams, 1990], [Guy, 2004],

$$P(n) = \left|47n^2 - 1701n + 10181\right| ,$$

generates 43 primes: 10181, 8527, 6967, 5501, 4129, 2851, 1667, 577, 419, 1321, 2129, 2843, 3463, 3989 4421, 4759, 5003, 5153, 5209, 5171, 5039, 4813, 4493, 4079, 3571, 2969, 2273, 1483, 599, 379, 1451, 2617, 3877, 5231, 6679, 8221, 9857, 11587, 13411, 15329, 17341, 19447, 21647, [Sloane, 2014, A050268].

17. Ruiz [2005] polynomial, $P(n) = \left|3n^3 - 183n^2 + 3318n - 18757\right|$, generates 43 primes: 18757, 15619, 12829, 10369, 8221, 6367, 4789, 3469, 2389, 1531, 877, 409, 109, 41, 59, 37, 229, 499, 829, 1201, 1597, 1999, 2389, 2749, 3061, 3307, 3469, 3529, 3469, 3271, 2917, 2389 1669, 739, 419, 1823, 3491, 5441, 7691, 10259, 13163, 16421, 20051, 24071, 28499, 33353, 38651 . There are 47 primes, but 2389 and 3469 are tripled, therefore it rests only 43 of distinct primes.

18. Fung and Ruby polynomial, [Fung and Williams, 1990], $P(n) = \left|36n^2 - 810n + 2753\right|$, generates 45 primes: 2753, 1979, 1277, 647, 89, 397, 811, 1153, 1423, 1621, 1747, 1801, 1783, 1693, 1531, 1297 991, 613, 163, 359, 953, 1619, 2357, 3167, 4049, 5003, 6029, 7127, 8297, 9539, 10853, 12239, 13697, 15227, 16829, 18503, 20249, 22067, 23957, 25919, 27953, 30059, 32237, 34487, 36809 .

19. Kazmenko and Trofimov polynomial, [Kazmenko and Trofimov, 2006]

$$P(n) = \left|-66n^3 + 3845n^2 - 60897n + 251831\right| ,$$

generates 46 primes: 251831, 194713, 144889, 101963, 65539, 35221, 10613, 8681, 23057, 32911, 38639, 40637, 39301, 35027, 28211, 19249 8537, 3529, 16553, 30139, 43891, 57413, 70309, 82183, 92639, 101281, 107713, 111539, 112363, 109789, 103421, 92863, 77719, 57593, 32089, 811, 36637, 80651, 131627, 189961, 256049, 330287, 413071, 504797, 605861, 716659 .

20. Wroblewski and Meyrignac polynomial, [Wroblewski and Meyrignac, 2006]

$$P(n) = \left|n^5 - 99n^4 + 3588n^3 - 56822n^2 + 348272n - 286397\right| ,$$

generates 47 primes: 286397, 8543, 210011, 336121, 402851, 424163, 412123, 377021, 327491, 270631, 212123, 156353, 106531, 64811, 32411, 9733, 3517, 8209, 5669, 2441, 14243, 27763, 41051, 52301, 59971, 62903, 60443, 52561, 39971, 24251, 7963, 5227, 10429, 1409, 29531, 91673, 196003, 355331, 584411, 900061, 1321283, 1869383, 2568091, 3443681, 4525091, 5844043, 7435163 .

21. Beyleveld [2006] polynomial,

$$P(n) = \left|n^4 - 97n^3 + 329n^2 - 45458n + 213589\right| ,$$

generates 49 primes: 213589, 171329, 135089, 104323, 78509, 57149, 39769, 25919, 15173, 7129, 1409, 2341, 4451, 5227, 4951, 3881, 2251, 271, 1873, 4019, 6029, 7789, 9209, 10223, 10789, 10889, 10529, 9739, 8573, 7109, 5449, 3719, 2069, 673, 271, 541, 109, 1949, 5273, 10399, 17669, 27449, 40129, 56123, 75869, 99829, 128489, 162359, 201973, 247889 . In the list, we have 50 primes, but the number 271 repeats once.

22. Wroblewski and Meyrignac polynomial, [Wroblewski and Meyrignac,

2006]

$$P(n)=\left|\frac{n^6-126n^5+6217n^4-153066n^3}{36}\right.$$
$$\left.+\frac{1987786n^2-1305531n+34747236}{36}\right|\ ,$$

generates 55 primes: 965201, 653687, 429409, 272563, 166693, 98321, 56597, 32969, 20873, 15443, 13241, 12007, 10429, 7933, 4493, 461, 3583, 6961, 9007, 9157, 7019, 2423, 4549, 13553, 23993, 35051, 45737, 54959, 61613, 64693, 63421, 57397, 46769, 32423, 16193, 1091, 8443, 6271, 15733, 67993, 163561, 318467, 552089, 887543, 1352093, 1977581, 2800877, 3864349, 5216353, 6911743, 9012401, 11587787, 14715509, 18481913, 22982693 .

23. Dress, Laudreau and Gupta polynomial, [Dress and Landreau, 2002, Gupta, 2006],

$$P(n)=\left|\frac{n^5-133n^4+6729n^3-158379n^2+1720294n-6823316}{4}\right|\ ,$$

generates 57 primes: 1705829, 1313701, 991127, 729173, 519643, 355049, 228581, 134077, 65993, 19373, 10181, 26539, 33073, 32687, 27847, 20611, 12659, 5323, 383, 3733, 4259, 1721, 3923, 12547, 23887, 37571, 53149, 70123, 87977, 106207, 124351, 142019, 158923, 174907, 189977, 204331, 218389, 232823, 248587, 266947, 289511, 318259, 355573, 404267, 467617, 549391, 653879, 785923, 950947, 1154987, 1404721, 1707499, 2071373, 2505127, 3018307, 3621251, 4325119 .

1.7 Primorial

Let p_n be the nth prime, then the *primorial* is defined by

$$p_n\#=\prod_{k=1}^{n}p_k\ . \tag{1.1}$$

By definition we have that $1\#=1$.

The values of $1\#$, $2\#$, $3\#$, …, $43\#$ are: 1, 2, 6, 30, 210, 2310, 30030, 510510, 9699690, 223092870, 6469693230, 200560490130, 7420738134810, 304250263527210, 13082761331670030, [Sloane, 2014, A002110] .

This list can also be obtained with the program 1.14, which generates the multiprimorial.

Program 1.14. for generating the multiprimorial, $p\# = kP(p,1)$ or $p\#\# = kP(p,2)$ or $p\#\#\# = kP(p,3)$.

```
kP(p,k) := | return 1 if p=1
           | return "Error. p not prime if TS(p)=0
           | q ← mod(p, k+1)
           | return p if q=0
           | pk ← 1
           | pk ← 2 if k=1
           | j ← 1
           | while prime_j ≤ p
           |   | pk ← pk · prime_j if mod(prime_j, k+1)=q
           |   | j ← j+1
           | return pk
```

The program calls the primality test *TS*, 1.5.

It is sometimes convenient to define the primorial $n\#$ for values other than just the primes, in which case it is taken to be given by the product of all primes less than or equal to n, i.e.

$$n\# = \prod_{k=1}^{\pi(n)} p_k\,, \tag{1.2}$$

where π is the prime counting function.

For 1, ..., 30 the first few values of $n\#$ are: 1, 2, 6, 6, 30, 30, 210, 210, 210, 210, 2310, 2310, 30030, 30030, 30030, 30030, 510510, 510510, 9699690, 9699690, 9699690, 9699690, 223092870, 223092870, 223092870, 223092870, 223092870, 223092870, 6469693230, 6469693230, [Sloane, 2014, A034386].

The decomposition in prime factors of numbers $prime_n\#-1$ and $prime_n\#+1$ for $n = 1,2,\ldots,15$ are in Tables 1.3 and respectively 1.4

Table 1.3: The decomposition in factors of $prime_n\#-1$

$prime_n\#-1$	*factorization*
0	0
1	1
5	5
29	29
209	$11\cdot 19$
2309	2309
30029	30029

Continued on next page

$prime_n\#-1$	*factorization*
510509	$61\cdot 8369$
9699689	$53\cdot 197\cdot 929$
223092869	$37\cdot 131\cdot 46027$
6469693229	$79\cdot 81894851$
200560490129	$228737\cdot 876817$
7420738134809	$229\cdot 541\cdot 1549\cdot 38669$
304250263527209	304250263527209
13082761331670029	$141269\cdot 92608862041$

Table 1.4: The decomposition in factors of $prime_n\#+1$

$prime_n\#+1$	*factorization*
2	2
3	3
7	7
31	31
211	211
2311	2311
30031	$59\cdot 509$
510511	$19\cdot 97\cdot 277$
9699691	$347\cdot 27953$
223092871	$317\cdot 703763$
6469693231	$331\cdot 571\cdot 34231$
200560490131	200560490131
7420738134811	$181\cdot 60611\cdot 676421$
304250263527211	$61\cdot 450451\cdot 11072701$
13082761331670031	$167\cdot 78339888213593$

1.7.1 Double Primorial

Let $p\in\mathbb{P}_{\geq 2}$, then the *double primorial* is defined by

$$p\#\# = p_1\cdot p_2\cdots p_m \ \text{ with } \quad \mod(p_j,3) = \quad \mod(p,3)\ , \tag{1.3}$$

where $p_j\in\mathbb{P}_{\geq 2}$ and $p_j<p$, for any $j=1,2,\ldots,m-1$ and $p_m=p$. By definition we have that $1\#\#=1$.

Examples: $2\#\# = 2$, $3\#\# = 3$, $5\#\# = 2\cdot 5 = 10$ because $\mod(5,3) = \mod(2,3) = 2$, $13\#\# = 7\cdot 13 = 91$ because $\mod(13,3) = \mod(7,3) = 1$, $17\#\# = 2\cdot 5\cdot 11\cdot 17 = 1870$ because $\mod(17,3) = \mod(11,3) = \mod(5,3) = \mod(2,3) = 2$, etc.

The list of values 1##, 2##, ..., 67##, obtained with the program kP, 1.14, is: 1, 2, 3, 10, 7, 110, 91, 1870, 1729, 43010, 1247290, 53599, 1983163, 51138890, 85276009, 2403527830, 127386974990, 7515831524410, 5201836549, 348523048783 .

The decomposition in prime factors of the numbers $prime_n\#\# - 1$ and $prime_n\#\# + 1$ for $n = 1, 2, \ldots, 19$ are in Tables 1.5 and respectively 1.6:

Table 1.5: The decomposition in factors of $prime_n\#\# - 1$

$prime_n\#\# - 1$	*factorization*
1	1
2	2
9	3^2
6	$2\cdot 3$
109	109
90	$2\cdot 3^2\cdot 5$
1869	$3\cdot 7\cdot 89$
1728	$2^6\cdot 3^3$
43009	$41\cdot 1049$
1247289	$3\cdot 379\cdot 1097$
53598	$2\cdot 3\cdot 8933$
1983162	$2\cdot 3\cdot 103\cdot 3209$
51138889	$67\cdot 763267$
85276008	$2^3\cdot 3^2\cdot 29\cdot 40841$
2403527829	$3\cdot 23537\cdot 34039$
127386974989	$19\cdot 59\cdot 113636909$
7515831524409	$3^2\cdot 13\cdot 101\cdot 19211\cdot 33107$
5201836548	$2^2\cdot 3\cdot 9277\cdot 46727$
348523048782	$2\cdot 3^3\cdot 23\cdot 280614371$

Table 1.6: The decomposition in factors of $prime_n\#\#+1$

$prime_n\#\#+1$	*factorization*
3	3
4	2^2
11	11
8	2^3
111	$3\cdot 37$
92	$2^2\cdot 23$
1871	1871
1730	$2\cdot 5\cdot 173$
43011	$3^6\cdot 59$
1247291	1247291
53600	$2^5\cdot 5^2\cdot 67$
1983164	$2^2\cdot 495791$
51138891	$3^3\cdot 1894033$
85276010	$2\cdot 5\cdot 8527601$
2403527831	$12889\cdot 186479$
127386974991	$3\cdot 349\cdot 121668553$
7515831524411	$7\cdot 1367\cdot 785435419$
5201836550	$2\cdot 5^2\cdot 104036731$
348523048784	$2^4\cdot 113\cdot 192767173$

1.7.2 Triple Primorial

Let $p\in\mathbb{P}_{\geq 2}$, then the *triple primorial* is defined by

$$p\#\#\# = p_1\cdot p_2\cdots p_m \ \text{ with } \quad \mod(p_j,4) = \quad \mod(p,4) \tag{1.4}$$

where $p_j\in\mathbb{P}_{\geq 2}$, $p_j<p$, for any $j=1,2,\ldots,m-1$ and $p_m=p$. By definition we have that $1\#\#\#=1$.

The list of values 1###, 2###, ..., 73###, obtained with the program *kP*, 1.14, is: 1, 2, 3, 5, 21, 231, 65, 1105, 4389, 100947, 32045, 3129357, 1185665, 48612265, 134562351, 6324430497, 2576450045, 373141399323, 157163452745, 25000473754641, 1775033636579511, 11472932050385 .

The decomposition in prime factors of $prime_n\#\#\#-1$ and $prime_n\#\#\#+1$ for $n=1,2,\ldots,19$ are in Tables 1.7 and respectively 1.8:

Table 1.7: The decomposition in factors of $prime_n\#\#\#-1$

$prime_n\#\#\#-1$	*factorization*
1	1
2	2
4	2^2
20	$2^2\cdot 5$
230	$2\cdot 5\cdot 23$
64	2^6
1104	$2^4\cdot 3\cdot 23$
4388	$2^2\cdot 1097$
100946	$2\cdot 17\cdot 2969$
32044	$2^2\cdot 8011$
3129356	$2^2\cdot 782339$
1185664	$2^7\cdot 59\cdot 157$
48612264	$2^3\cdot 3\cdot 2025511$
134562350	$2\cdot 5^2\cdot 13\cdot 241\cdot 859$
6324430496	$2^5\cdot 197638453$
2576450044	$2^2\cdot 7\cdot 263\cdot 349871$
373141399322	$2\cdot 186570699661$
157163452744	$2^3\cdot 193\cdot 421\cdot 241781$
25000473754640	$2^4\cdot 5\cdot 312505921933$

Table 1.8: The decomposition in factors of $prime_n\#\#\#+1$

$prime_n\#\#\#+1$	*factorization*
2	2
3	3
4	2^2
6	$2\cdot 3$
22	$2\cdot 11$
232	$2^3\cdot 29$
66	$2\cdot 3\cdot 11$
1106	$2\cdot 7\cdot 79$
4390	$2\cdot 5\cdot 439$
100948	$2^2\cdot 25237$
32046	$2\cdot 3\cdot 7^2\cdot 109$
3129358	$2\cdot 1564679$

Continued on next page

$prime_n\#\#\# + 1$	*factorization*
1185666	$2 \cdot 3 \cdot 73 \cdot 2707$
48612266	$2 \cdot 131 \cdot 185543$
134562352	$2^4 \cdot 1123 \cdot 7489$
6324430498	$2 \cdot 37 \cdot 6269 \cdot 13633$
2576450046	$2 \cdot 3 \cdot 19 \cdot 61 \cdot 163 \cdot 2273$
373141399324	$2^2 \cdot 433 \cdot 215439607$
157163452746	$2 \cdot 3 \cdot 5647 \cdot 4638553$
25000473754642	$2 \cdot 109 \cdot 6199 \cdot 18499931$

Chapter 2

Arithmetical Functions

2.1 Function of Counting the Digits

Mathcad user functions required in the following.

Function 2.1. Function of counting the digits of number $n_{(10)}$ in base b:

$$nrd(n,b) := \left| \begin{array}{l} return\ \ 1\ \ if\ \ n{=}0 \\ return\ \ 1 + floor\big(\log(n,b)\big)\ \ otherwise \end{array} \right.$$

2.2 Digits the Number in Base b

Program 2.2. Program providing the digits in base b of the number $n_{(10)}$:

$$dn(n,b) := \left| \begin{array}{l} for\ \ k \in ORIGIN..nrd(n,b)-1 \\ \quad \left| \begin{array}{l} t \leftarrow Trunc(n,b) \\ cb_k \leftarrow n - t \\ n \leftarrow \dfrac{t}{b} \end{array} \right. \\ return\ \ reverse(cb) \end{array} \right.$$

2.3 Prime Counting Function

By means of Smarandache's function, we obtain a formula for counting the prime numbers less or equal to n, [Seagull, 1995].

Program 2.3. of counting the primes to n.

$$\pi(n) := \left| \begin{array}{l} return\ \ 0\ \ if\ \ n = 1 \\ return\ \ 1\ \ if\ \ n = 2 \end{array} \right.$$

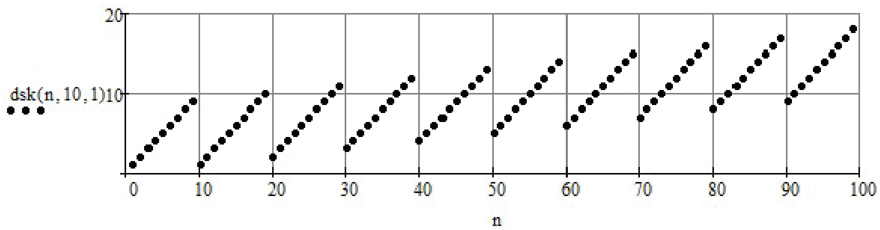

Figure 2.1: The digital sum function

$$\left| \begin{array}{l} return\ \ 2\ \ if\ \ n = 3 \\ return\ \ -1 + \displaystyle\sum_{k=2}^{n} floor\left(\frac{S_k}{k}\right) \end{array} \right.$$

2.4 Digital Sum

Function 2.4. Function of summing the digits in base b of power k of the number n written in base 10.

$$dks(n, b, k) := \sum \overrightarrow{dn(n, b)}^{k} . \tag{2.1}$$

Examples:

1. Example $dks(76, 8, 1) = 6$, verified by the identity $76_{(10)} = 114_{(8)}$ and by the fact than $1^1 + 1^1 + 4^1 = 6$;

2. Example $dks(1234, 16, 1) = 19$, verified by the identity $1234_{(10)} = 4d2_{(16)}$ and by the fact than $4^1 + d^1 + 2^1 = 4 + 13 + 2 = 19$;

3. Example $dks(15, 2, 1) = 4$, verified by the identity $15_{(10)} = 1111_{(2)}$ and by the fact than $1^1 + 1^1 + 1^1 + 1^1 = 4$.

4. Example $dks(76, 8, 2) = 18$, verified by the identity $76_{(10)} = 114_{(8)}$ and by the fact than $1^2 + 1^2 + 4^2 = 18$;

5. Example $dks(1234, 16, 2) = 189$, verified by the identity $1234_{(10)} = 4d2_{(16)}$ and by the fact than $4^2 + d^2 + 2^2 = 4^2 + 13^2 + 2^2 = 189$;

6. Example $dks(15, 2) = 4$, verified by the identity $15_{(10)} = 1111_{(2)}$ and by the fact than $1^2 + 1^2 + 1^2 + 1^2 = 4$.

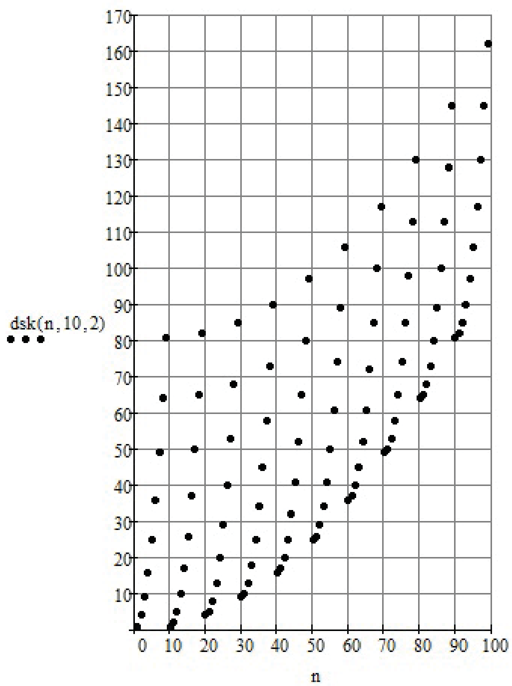

Figure 2.2: Digital sum function of power 2 of the number $n_{(10)}$

2.4.1 Narcissistic Numbers

We can apply the function *dks*, given by (2.1), for determining Narcissistic numbers (Armstrong numbers, or Plus Perfect numbers), [Weisstein, 2014d], [Sloane, 2014, A005188, A003321, A010344, A010346, A010348, A010350, A010353, A010354, A014576, A023052, A032799, A046074, A101337 and A11490], [Hardy, 1993, p. 105], [Madachy, 1979, pp. 163–173], [Roberts, 1992, p. 35], [Pickover, 1995, pp. 169–170], [Pickover, 2001, pp. 204–205].

Definition 2.5. A number having m digits d_k in base b ($b \in \mathbb{N}^*$, $b \geq 2$) is Narcissistic if

$$\overline{d_1 d_2 \ldots d_m}_{(b)} = d_1^m + d_2^m + \ldots + d_m^m \,, \tag{2.2}$$

where $d_k \in \{0, 1, \ldots, b-1\}$, for $k \in I_m = \{1, 2, \ldots, m\}$.

Program 2.2 can be used for determining Narcissistic numbers in numeration base b, [Cira and Cira, 2010]. The numbers $\{1, 2, \ldots, b-1\}$ are ordinary Narcissistic numbers for any $b \geq 2$.

To note that the search domain for a base $b \in \mathbb{N}$, $b \geq 2$ are finite. For any number n, with m digits in base b for which we have met the condition

$\log_b\left(m(b-1)^m\right) > \log_b(b^{m-1})$ Narcissistic number search makes no sense. For example, the search domain for numbers in base 3 makes sense only for the numbers: 3, 4, ..., 2048.

Let the function $h:\mathbb{N}^* \times \mathbb{N}^* \to \mathbb{R}$ given by the formula

$$h(b,m) = \log_b\left(m(b-1)^m\right) - \log_b(b^{m-1}) \ . \tag{2.3}$$

We represent the function for $b = 3, 4, \ldots, 16$ and $m = 1, 2, \ldots, 120$, see Figures 2.3. Using the function h we can determine the Narcissistic numbers' search domains.

Let the search domains be:

$$Dc_2 = \{2\}\,, \tag{2.4}$$
$$Dc_3 = \{3, 4, \ldots, 2048\}\,,\ \text{where } 2048 = 8 \cdot 2^8, \tag{2.5}$$
$$Dc_b = \left\{b, b+1, \ldots, 10^7\right\} \ \text{ for } \ b = \{4, 5, \ldots, 16\} \ . \tag{2.6}$$

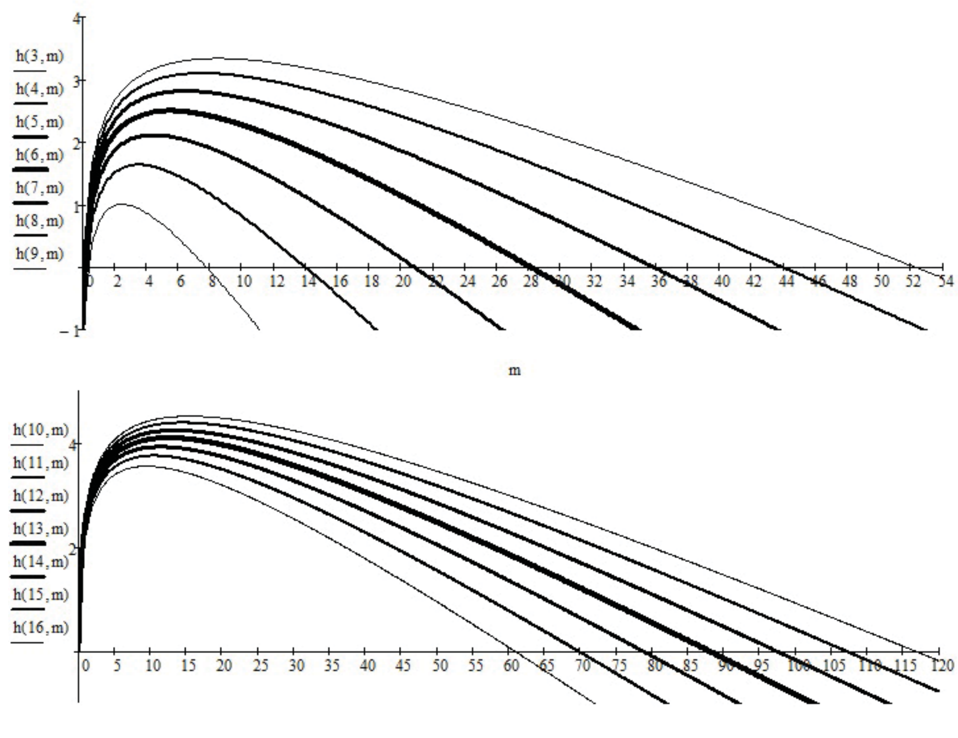

Figure 2.3: Function h for $b = 3, 4, \ldots, 16$ and $m = 1, 2, \ldots, 120$

Table 2.1: Narcissistic numbers with $b=3$ of (2.5)

$5=$	$12_{(3)}=$	1^2+2^2
$8=$	$22_{(3)}=$	2^2+2^2
$17=$	$122_{(3)}=$	$1^3+2^3+2^3$

Table 2.2: Narcissistic numbers with $b=4$ of (2.6)

$28=$	$130_{(4)}=$	$1^3+3^3+0^3$
$29=$	$131_{(4)}=$	$1^3+3^3+1^3$
$35=$	$203_{(4)}=$	$2^3+0^3+3^3$
$43=$	$223_{(4)}=$	$2^3+2^3+3^3$
$55=$	$313_{(4)}=$	$3^3+1^3+3^3$
$62=$	$332_{(4)}=$	$3^3+3^3+2^3$
$83=$	$1103_{(4)}=$	$1^4+1^4+0^4+3^4$
$243=$	$3303_{(4)}=$	$3^4+3^4+0^4+3^4$

Table 2.3: Narcissistic numbers with $b=5$ of (2.6)

$13=$	$23_{(5)}=$	2^2+3^2
$18=$	$33_{(5)}=$	3^2+3^2
$28=$	$103_{(5)}=$	$1^3+0^3+3^3$
$118=$	$433_{(5)}=$	$4^3+3^3+3^3$
$289=$	$2124_{(5)}=$	$2^4+1^4+2^4+4^4$
$353=$	$2403_{(5)}=$	$2^4+4^4+0^4+3^4$
$419=$	$3134_{(5)}=$	$3^4+1^4+3^4+3^4$
$4890=$	$124030_{(5)}=$	$1^6+2^6+4^6+0^6+3^6+0^6$
$4891=$	$124031_{(5)}=$	$1^6+2^6+4^6+0^6+3^6+1^6$
$9113=$	$242423_{(5)}=$	$2^6+4^6+2^6+4^6+2^6+3^6$

Table 2.4: Narcissistic numbers with $b=6$ of (2.6)

$99=$	$243_{(6)}=$	$2^3+4^3+3^3$

Continued on next page

$190 =$	$514_{(6)} =$	$5^3+1^3+4^3$
$2292 =$	$14340_{(6)} =$	$1^5+4^5+3^5+4^5+0^5$
$2293 =$	$14341_{(6)} =$	$1^5+4^5+3^5+4^5+1^5$
$2324 =$	$14432_{(6)} =$	$1^5+4^5+4^5+3^5+2^5$
$3432 =$	$23520_{(6)} =$	$2^5+3^5+5^5+2^5+0^5$
$3433 =$	$23521_{(6)} =$	$2^5+3^5+5^5+2^5+1^5$
$6197 =$	$44405_{(6)} =$	$4^5+4^5+4^5+0^5+5^5$
$36140 =$	$435152_{(6)} =$	$4^6+3^6+5^6+1^6+5^6+2^6$
$269458 =$	$5435254_{(6)} =$	$5^7+4^7+3^7+5^7+2^7+5^7+4^7$
$391907 =$	$12222215_{(6)} =$	$1^8+2^8+2^8+2^8+2^8+2^8+1^8+5^8$

Table 2.5: Narcissistic numbers with $b=7$ of (2.6)

$10 =$	$13_{(7)} =$	1^2+3^2
$25 =$	$34_{(7)} =$	3^2+4^2
$32 =$	$44_{(7)} =$	4^2+4^2
$45 =$	$63_{(7)} =$	6^2+3^2
$133 =$	$250_{(7)} =$	$2^3+5^3+0^3$
$134 =$	$251_{(7)} =$	$2^3+5^3+1^3$
$152 =$	$305_{(7)} =$	$3^3+0^3+5^3$
$250 =$	$505_{(7)} =$	$5^3+0^3+5^3$
$3190 =$	$12205_{(7)} =$	$1^5+2^5+2^5+0^5+5^5$
$3222 =$	$12252_{(7)} =$	$1^5+2^5+2^5+5^5+2^5$
$3612 =$	$13350_{(7)} =$	$1^5+3^5+3^5+5^5+0^5$
$3613 =$	$13351_{(7)} =$	$1^5+3^5+3^5+5^5+1^5$
$4183 =$	$15124_{(7)} =$	$1^5+5^5+1^5+2^5+4^5$
$9286 =$	$36034_{(7)} =$	$3^5+6^5+0^5+3^5+4^5$
$35411 =$	$205145_{(7)} =$	$2^6+0^6+5^6+1^6+4^6+5^6$
$191334 =$	$1424553_{(7)} =$	$1^7+4^7+2^7+4^7+5^7+5^7+3^7$
$193393 =$	$1433554_{(7)} =$	$1^7+4^7+3^7+3^7+5^7+5^7+4^7$
$376889 =$	$3126542_{(7)} =$	$3^7+1^7+2^7+6^7+5^7+4^7+2^7$
$535069 =$	$4355653_{(7)} =$	$4^7+3^7+5^7+5^7+6^7+5^7+3^7$
$794376 =$	$6515652_{(7)} =$	$6^7+5^7+1^7+5^7+6^7+5^7+2^7$

Table 2.6: Narcissistic numbers with $b = 8$ of (2.6)

$20 =$	$24_{(8)} =$	$2^2 + 4^2$
$52 =$	$64_{(8)} =$	$6^2 + 4^2$
$92 =$	$134_{(8)} =$	$1^3 + 3^3 + 4^3$
$133 =$	$205_{(8)} =$	$2^3 + 0^3 + 5^3$
$307 =$	$463_{(8)} =$	$4^3 + 6^3 + 3^3$
$432 =$	$660_{(8)} =$	$6^3 + 6^3 + 0^3$
$433 =$	$661_{(8)} =$	$6^3 + 6^3 + 1^3$
$16819 =$	$40663_{(8)} =$	$4^5 + 0^5 + 6^5 + 6^5 + 3^5$
$17864 =$	$42710_{(8)} =$	$4^5 + 2^5 + 7^5 + 1^5 + 0^5$
$17865 =$	$42711_{(8)} =$	$4^5 + 2^5 + 7^5 + 1^5 + 1^5$
$24583 =$	$60007_{(8)} =$	$6^5 + 0^5 + 0^5 + 0^5 + 7^5$
$25639 =$	$62047_{(8)} =$	$6^5 + 2^5 + 0^5 + 4^5 + 7^5$
$212419 =$	$636703_{(8)} =$	$6^6 + 3^6 + 6^6 + 7^6 + 0^6 + 3^6$
$906298 =$	$3352072_{(8)} =$	$3^7 + 3^7 + 5^7 + 2^7 + 0^7 + 7^7 + 2^7$
$906426 =$	$3352272_{(8)} =$	$3^7 + 3^7 + 5^7 + 2^7 + 2^7 + 7^7 + 2^7$
$938811 =$	$3451473_{(8)} =$	$3^7 + 4^7 + 5^7 + 1^7 + 4^7 + 7^7 + 3^7$

Table 2.7: Narcissistic numbers with $b = 9$ of (2.6)

$41 =$	$45_{(9)} =$	$4^2 + 5^2$
$50 =$	$55_{(9)} =$	$5^2 + 5^2$
$126 =$	$150_{(9)} =$	$1^3 + 5^3 + 0^3$
$127 =$	$151_{(9)} =$	$1^3 + 5^3 + 1^3$
$468 =$	$570_{(9)} =$	$5^3 + 7^3 + 0^3$
$469 =$	$571_{(9)} =$	$5^3 + 7^3 + 1^3$
$1824 =$	$2446_{(9)} =$	$2^4 + 4^4 + 4^4 + 6^4$
$8052 =$	$12036_{(9)} =$	$1^5 + 2^5 + 0^5 + 3^5 + 6^5$
$8295 =$	$12336_{(9)} =$	$1^5 + 2^5 + 3^5 + 3^5 + 6^5$
$9857 =$	$14462_{(9)} =$	$1^5 + 4^5 + 4^5 + 6^5 + 2^5$

Table 2.8: Narcissistic numbers with $b = 10$ of (2.6)

$153 =$	$1^3 + 5^3 + 3^3$

Continued on next page

370 =	$3^3+7^3+0^3$
371 =	$3^3+7^3+1^3$
407 =	$4^3+0^3+7^3$
1634 =	$1^4+6^4+3^4+4^4$
8208 =	$8^4+2^4+0^4+8^4$
9474 =	$9^4+4^4+7^4+4^4$
54748 =	$5^5+4^5+7^5+4^5+8^5$
92727 =	$9^5+2^5+7^5+2^5+7^5$
93084 =	$9^5+3^5+0^5+8^5+4^5$
548834 =	$5^6+4^6+8^6+8^6+3^6+4^6$

The list of all Narcissistic numbers in base 10 is known, [Weisstein, 2014d]. Besides those in Table 2.8 we still have the following Narcissistic numbers: 1741725, 4210818, 9800817, 9926315, 24678050, 24678051, 88593477, 146511208, 472335975, 534494836, 912985153, 4679307774, 32164049650, 32164049651, ... [Sloane, 2014, A005188].

The biggest Narcissistic numbers have 39 digits; they are:

$$115132219018763992565095597973971522400,\\ 115132219018763992565095597973971522401\ .$$

Observation 2.6. As known, the digits of numeration base 16 are: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 denoted respectively by a, 11 with b, 12 with c, 13 with d, 14 with e and 15 with f. Naturally, for bases bigger than 16 we use the following digits notation: $16=g$, $17=h$, $18=i$, $19=j$, $20=k$, $21=\ell$, $22=m$, $23=n$, $24=o$, $25=p$, $26=q$, $27=r$, $28=s$, $29=t$, $30=u$, $31=v$, $32=w$, $33=x$, $34=y$, $35=z$, $36=A$, $37=B$, $38=C$, $39=D$, $40=E$, $41=F$, $42=G$, $43=H$, $44=I$, ...

Table 2.9: Narcissistic numbers with $b=11$ of (2.6)

61 =	$56_{(11)}$ =	5^2+6^2
72 =	$66_{(11)}$ =	6^2+6^2
126 =	$105_{(11)}$ =	$1^3+0^3+5^3$
370 =	$307_{(11)}$ =	$3^3+0^3+7^3$
855 =	$708_{(11)}$ =	$7^3+0^3+8^3$
1161 =	$966_{(11)}$ =	$9^3+6^3+6^3$
1216 =	$a06_{(11)}$ =	$10^3+0^3+6^3$
1280 =	$a64_{(11)}$ =	$10^3+6^3+4^3$

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$10657 =$	$8009_{(11)} =$	$8^4+0^4+0^4+9^4$
$16841 =$	$11720_{(11)} =$	$1^5+1^5+7^5+2^5+0^5$
$16842 =$	$11721_{(11)} =$	$1^5+1^5+7^5+2^5+1^5$
$17864 =$	$12470_{(11)} =$	$1^5+2^5+4^5+7^5+0^5$
$17865 =$	$12471_{(11)} =$	$1^5+2^5+4^5+7^5+1^5$
$36949 =$	$25840_{(11)} =$	$2^5+5^5+8^5+4^5+0^5$
$36950 =$	$25841_{(11)} =$	$2^5+5^5+8^5+4^5+1^5$
$63684 =$	$43935_{(11)} =$	$4^5+3^5+9^5+3^5+5^5$
$66324 =$	$45915_{(11)} =$	$4^5+5^5+9^5+1^5+5^5$
$71217 =$	$49563_{(11)} =$	$4^5+9^5+5^5+6^5+3^5$
$90120 =$	$61788_{(11)} =$	$6^5+1^5+7^5+8^5+8^5$
$99594 =$	$68910_{(11)} =$	$6^5+8^5+9^5+1^5+0^5$
$99595 =$	$68911_{(11)} =$	$6^5+8^5+9^5+1^5+1^5$
$141424 =$	$97288_{(11)} =$	$9^5+7^5+2^5+8^5+8^5$
$157383 =$	$a8276_{(11)} =$	$10^5+8^5+2^5+7^5+6^5$

Table 2.10: Narcissistic numbers with $b=12$ of (2.6)

$29 =$	$25_{(12)} =$	2^2+5^2
$125 =$	$a5_{(12)} =$	10^2+5^2
$811 =$	$577_{(12)} =$	$5^3+7^3+7^3$
$944 =$	$668_{(12)} =$	$6^3+6^3+8^3$
$1539 =$	$a83_{(12)} =$	$10^3+8^3+3^3$
$28733 =$	$14765_{(12)} =$	$1^5+4^5+7^5+6^5+5^5$
$193084 =$	$938a4_{(12)} =$	$9^5+3^5+8^5+10^5+4^5$
$887690 =$	$369862_{(12)} =$	$3^6+6^6+9^6+8^6+6^6+2^6$

Table 2.11: Narcissistic numbers with $b=13$ of (2.6)

$17 =$	$14_{(13)} =$	1^2+4^2
$45 =$	$36_{(13)} =$	3^2+6^2
$85 =$	$67_{(13)} =$	6^2+7^2
$98 =$	$77_{(13)} =$	7^2+7^2
$136 =$	$a6_{(13)} =$	10^2+6^2
$160 =$	$c4_{(13)} =$	12^2+4^2

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$793 =$	$490_{(13)} =$	$4^3+9^3+0^3$
$794 =$	$491_{(13)} =$	$4^3+9^3+1^3$
$854 =$	$509_{(13)} =$	$5^3+0^3+9^3$
$1968 =$	$b85_{(13)} =$	$11^3+8^3+5^3$
$8194 =$	$3964_{(13)} =$	$3^4+9^4+6^4+4^4$
$62481 =$	$22593_{(13)} =$	$2^5+2^5+5^5+9^5+3^5$
$167544 =$	$5b350_{(13)} =$	$5^5+11^5+3^5+5^5+0^5$
$167545 =$	$5b351_{(13)} =$	$5^5+11^5+3^5+5^5+1^5$
$294094 =$	$a3b28_{(13)} =$	$10^5+3^5+11^5+2^5+8^5$
$320375 =$	$b2a93_{(13)} =$	$11^5+2^5+10^5+9^5+3^5$
$323612 =$	$b43b3_{(13)} =$	$11^5+4^5+3^5+11^5+3^5$
$325471 =$	$b51b3_{(13)} =$	$11^5+5^5+1^5+11^5+3^5$
$325713 =$	$b533b_{(13)} =$	$11^5+5^5+3^5+3^5+11^5$
$350131 =$	$c34a2_{(13)} =$	$12^5+3^5+4^5+10^5+2^5$
$365914 =$	$ca723_{(13)} =$	$12^5+10^5+7^5+2^5+3^5$

Table 2.12: Narcissistic numbers with $b = 14$ of (2.6)

$244 =$	$136_{(14)} =$	$1^3+3^3+6^3$
$793 =$	$409_{(14)} =$	$4^3+0^3+9^3$
$282007 =$	$74ab5_{(14)} =$	$7^5+4^5+10^5+11^5+5^5$

Table 2.13: Narcissistic numbers with $b = 15$ of (2.6)

$113 =$	$78_{(15)} =$	7^2+8^2
$128 =$	$88_{(15)} =$	8^2+8^2
$2755 =$	$c3a_{(15)} =$	$12^3+3^3+10^3$
$3052 =$	$d87_{(15)} =$	$13^3+8^3+7^3$
$5059 =$	$1774_{(15)} =$	$1^4+7^4+7^4+4^4$
$49074 =$	$e819_{(15)} =$	$14^4+8^4+1^4+9^4$
$49089 =$	$e829_{(15)} =$	$14^4+8^4+2^4+9^4$
$386862 =$	$7995c_{(15)} =$	$7^5+9^5+9^5+5^5+12^5$
$413951 =$	$829bb_{(15)} =$	$8^5+2^5+9^5+11^5+11^5$
$517902 =$	$a36bc_{(15)} =$	$10^5+3^5+6^5+11^5+12^5$

Table 2.14: Narcissistic numbers with $b = 16$ of (2.6)

$342 =$	$156_{(16)} =$	$1^3+5^3+6^3$
$371 =$	$173_{(16)} =$	$1^3+7^3+3^3$
$520 =$	$208_{(16)} =$	$2^3+0^3+8^3$
$584 =$	$248_{(16)} =$	$2^3+4^3+8^3$
$645 =$	$285_{(16)} =$	$2^3+8^3+5^3$
$1189 =$	$4a5_{(16)} =$	$4^3+10^3+5^3$
$1456 =$	$5b0_{(16)} =$	$5^3+11^3+0^3$
$1457 =$	$5b1_{(16)} =$	$5^3+11^3+1^3$
$1547 =$	$60b_{(16)} =$	$6^3+0^3+11^3$
$1611 =$	$64b_{(16)} =$	$6^3+4^3+11^3$
$2240 =$	$8c0_{(16)} =$	$8^3+12^3+0^3$
$2241 =$	$8c1_{(16)} =$	$8^3+12^3+1^3$
$2458 =$	$99a_{(16)} =$	$9^3+9^3+10^3$
$2729 =$	$aa9_{(16)} =$	$10^3+10^3+9^3$
$2755 =$	$ac3_{(16)} =$	$10^3+12^3+3^3$
$3240 =$	$ca8_{(16)} =$	$12^3+10^3+8^3$
$3689 =$	$e69_{(16)} =$	$14^3+6^3+9^3$
$3744 =$	$ea0_{(16)} =$	$14^3+10^3+0^3$
$3745 =$	$ea1_{(16)} =$	$14^3+10^3+1^3$
$47314 =$	$b8d2_{(16)} =$	$11^4+8^4+13^4+2^4$
$79225 =$	$13579_{(16)} =$	$1^5+3^5+5^5+7^5+9^5$
$177922 =$	$2b702_{(16)} =$	$2^5+11^5+7^5+0^5+2^5$
$177954 =$	$2b722_{(16)} =$	$2^5+11^5+7^5+2^5+2^5$
$368764 =$	$5a07c_{(16)} =$	$5^5+10^5+0^5+7^5+12^5$
$369788 =$	$5a47c_{(16)} =$	$5^5+10^5+4^5+7^5+12^5$
$786656 =$	$c00e0_{(16)} =$	$12^5+0^5+0^5+14^5+0^5$
$786657 =$	$c00e1_{(16)} =$	$12^5+0^5+0^5+14^5+1^5$
$787680 =$	$c04e0_{(16)} =$	$12^5+0^5+4^5+14^5+0^5$
$787681 =$	$c04e1_{(16)} =$	$12^5+0^5+4^5+14^5+1^5$
$811239 =$	$c60e7_{(16)} =$	$12^5+6^5+0^5+14^5+7^5$
$812263 =$	$c64e7_{(16)} =$	$12^5+6^5+4^5+14^5+7^5$
$819424 =$	$c80e0_{(16)} =$	$12^5+8^5+0^5+14^5+0^5$
$819425 =$	$c80e1_{(16)} =$	$12^5+8^5+0^5+14^5+1^5$
$820448 =$	$c84e0_{(16)} =$	$12^5+8^5+4^5+14^5+0^5$
$820449 =$	$c84e1_{(16)} =$	$12^5+8^5+4^5+14^5+1^5$
$909360 =$	$de030_{(16)} =$	$13^5+14^5+0^5+3^5+0^5$

Continued on next page

$909361 =$	$de031_{(16)} =$	$13^5+14^5+0^5+3^5+1^5$
$910384 =$	$de430_{(16)} =$	$13^5+14^5+4^5+3^5+0^5$
$910385 =$	$de431_{(16)} =$	$13^5+14^5+4^5+3^5+1^5$
$964546 =$	$eb7c2_{(16)} =$	$14^5+11^5+7^5+12^5+2^5$

The following numbers are Narcissistic in two different numeration bases:

Table 2.15: Narcissistic numbers in two bases

$n_{(10)}$	$n_{(b_1)}$	$n_{(b_2)}$
17	$122_{(3)}$	$14_{(13)}$
28	$130_{(4)}$	$103_{(5)}$
29	$131_{(4)}$	$25_{(12)}$
45	$63_{(7)}$	$36_{(13)}$
126	$150_{(9)}$	$105_{(11)}$
133	$250_{(7)}$	$205_{(8)}$
370	$370_{(10)}$	$307_{(11)}$
371	$371_{(10)}$	$173_{(16)}$
793	$490_{(13)}$	$409_{(14)}$
2755	$c3a_{(15)}$	$ac3_{(16)}$
17864	$42710_{(8)}$	$12470_{(11)}$
17865	$42711_{(8)}$	$12471_{(11)}$

The question is whether there are multiple Narcissistic numbers in different numeration bases. There is the number $10261_{(10)}$ which is triple Narcissistic, in numeration bases $b = 31, 32$ and 49, indeed

$$\begin{aligned}
10261_{(10)} &= al0_{(31)} &&= 10^3+21^3+0^3\,,\\
10261_{(10)} &= a0\ell_{(32)} &&= 10^3+0^3+21^3\,,\\
10261_{(10)} &= 4dk_{(49)} &&= 4^3+13^3+20^3\,.
\end{aligned}$$

If we consider the numeration bases less or equal to 100 we have 3 triple Narcissistic numbers, in different numeration bases:

$n_{(10)}$	$n_{(b_1)}$	$n_{(b_2)}$	$n_{(b_3)}$
125	$a5_{(12)}$	$5a_{(23)}$	$2b_{(57)}$
2080	$Ic_{(47)}$	$As_{(57)}$	$sA_{(73)}$
10261	$a\ell 0_{(31)}$	$a0\ell_{(32)}$	$4dk_{(49)}$

where $k = 20$, $\ell = 21$, $s = 28$, $A = 36$ and $I = 44$.

To note that for $n \in \mathbb{N}^*$, $n \leq 10^6$ we don't have Narcissistic numbers in four different bases, where $b \in \mathbb{N}^*$ and $2 \leq b \leq 100$.

2.4.2 Inverse Narcissistic Numbers

Definition 2.7. A number in base b ($b \in \mathbb{N}^*$, $b \geq 2$) is *inverse Narcissistic* if

$$\overline{d_1 d_2 \ldots d_m}_{(b)} = m^{d_1} + m^{d_2} + \ldots + m^{d_m} , \tag{2.7}$$

where $d_k \in \{0, 1, \ldots, b-1\}$, for $k \in I_m$.

Let

$$\log_b(b^{m-1}) = \log_b\left(m^b\right) . \tag{2.8}$$

Lemma 2.8. *The numbers $n_{(b)}$ with the property* (2.7) *can't have more than n_d digits, where $n_d = \lceil s \rceil$, and s is the solution equation* (2.8).

Proof. The smallest number in base b with m digits is b^{m-1}. The biggest number with property (2.7) is $m \cdot m^{b-1}$. If the smallest number in base b with m digits is bigger than the biggest number with property (2.7), i.e. $b^{m-1} \geq m^b$, then this inequation provides the limit of numbers which can meet the condition (2.7).

If we logarithmize both terms of the inequation, we get an inequation which establishes the limit of possible digits number. This leads to solving the equation (2.8). Let s be the solution of the equation (2.8), but keeping into account that the digits number of a number is an integer resulting that $n_d = \lceil s \rceil$. □

Corollary 2.9. *The maximum number of digits of the numbers in base b, which meet the condition* (2.8) *are given in the Table* 2.16

b	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
n_d	7	6	7	8	8	9	10	11	12	13	14	15	16	17	18

Table 2.16: The maximum number of digits of the numbers in base b

Let the search domain be defined by

$$Dc_b = \left\{b, b+1, \ldots, n_d^b\right\} , \tag{2.9}$$

where n_d^b is not bigger than 10^7, and n_d are given in the Table (2.16). We avoid single digit numbers 0, 1, 2, …, $b-1$, because $1 = 1^1$ is a trivial solution, and $0 \neq 1^0$, $2 \neq 1^2$, … $b-1 \neq 1^{b-1}$, for any base b.

Therefore, the search domain are:

$$
\begin{aligned}
Dc_2 &= \{2,3,\ldots,49\}\,,\ \text{where } 49 = 7^2, && (2.10)\\
Dc_3 &= \{3,4,\ldots,216\}\,,\ \text{where } 216 = 6^3, && (2.11)\\
Dc_4 &= \{4,5,\ldots,2401\}\,,\ \text{where } 2401 = 7^4, && (2.12)\\
Dc_5 &= \{5,6,\ldots,32768\}\,,\ \text{where } 32768 = 8^5, && (2.13)\\
Dc_6 &= \{6,7,\ldots,262144\}\,,\ \text{where } 262144 = 8^6, && (2.14)\\
Dc_7 &= \{7,8,\ldots,4782969\}\,,\ \text{where } 4782969 = 9^7, && (2.15)
\end{aligned}
$$

and for $b = 8,9,\ldots,16$,

$$Dc_b = \left\{b, b+1,\ldots,10^7\right\} \text{for } \ b = 8,9,\ldots,16\,. \tag{2.16}$$

We determined all the inverse Narcissistic numbers in numeration bases $b = 2,3,\ldots,16$ on the search domains (2.10 – 2.16).

Table 2.17: Inverse narcissistic numbers of (2.10 – 2.16)

$10 =$	$1010_{(2)} =$	$4^1+4^0+4^1+4^0$
$13 =$	$1101_{(2)} =$	$4^1+4^1+4^0+4^1$
$3 =$	$10_{(3)} =$	2^1+2^0
$4 =$	$11_{(3)} =$	2^1+2^1
$8 =$	$22_{(3)} =$	2^2+2^2
$6 =$	$12_{(4)} =$	2^1+2^2
$39 =$	$213_{(4)} =$	$3^2+3^1+3^3$
$33 =$	$113_{(5)} =$	$3^1+3^1+3^3$
$117 =$	$432_{(5)} =$	$3^4+3^3+3^2$
$57 =$	$133_{(6)} =$	$3^1+3^3+3^3$
$135 =$	$343_{(6)} =$	$3^3+3^4+3^3$
$340 =$	$1324_{(6)} =$	$4^1+4^3+4^2+4^4$
$3281 =$	$23105_{(6)} =$	$5^2+5^3+5^1+5^0+5^5$
$10 =$	$13_{(7)} =$	2^1+2^3
$32 =$	$44_{(7)} =$	2^4+2^4
$245 =$	$500_{(7)} =$	$3^5+3^0+3^0$
$261 =$	$522_{(7)} =$	$3^5+3^2+3^2$
$20 =$	$24_{(8)} =$	2^2+2^4
$355747 =$	$1266643_{(8)} =$	$7^1+7^2+7^6+7^6+7^6+7^4+7^3$
$85 =$	$104_{(9)} =$	$3^1+3^0+3^4$
$335671 =$	$561407_{(9)} =$	$6^5+6^6+6^1+6^4+6^0+6^7$
$840805 =$	$1521327_{(9)} =$	$7^1+7^5+7^2+7^1+7^3+7^2+7^7$

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$842821 =$	$1524117_{(9)} =$	$7^1+7^5+7^2+7^4+7^1+7^1+7^7$
$845257 =$	$1527424_{(9)} =$	$7^1+7^5+7^2+7^7+7^4+7^2+7^4$
$4624 =$	$4624_{(10)} =$	$4^4+4^6+4^2+4^4$
$68 =$	$62_{(11)} =$	2^6+2^8
$16385 =$	$11346_{(11)} =$	$5^1+5^1+5^3+5^4+5^6$
	—	
	—	
$18 =$	$14_{(14)} =$	2^1+2^4
$93905 =$	$26317_{(14)} =$	$5^2+5^6+5^3+5^1+5^7$
$156905 =$	$41277_{(14)} =$	$5^4+5^1+5^2+5^7+5^7$
$250005 =$	$67177_{(14)} =$	$5^6+5^7+5^1+5^7+5^7$
	—	
$4102 =$	$1006_{(16)} =$	$4^1+4^0+4^0+4^6$

2.4.3 Münchhausen Numbers

Münchhausen numbers are a subchapter of Visual Representation Numbers, [Madachy, 1979, pp. 163–175], [Pickover, 1995, pp. 169–171], [Pickover, 2001], [Sloane, 2014, A046253], [Weisstein, 2014c].

Definition 2.10. A number in base b ($b \in \mathbb{N}^*$, $b \geq 2$) is *Münchhausen number* if

$$\overline{d_1 d_2 \ldots d_m}_{(b)} = d_1^{d_1} + d_2^{d_2} + \ldots + d_m^{d_m} \,, \tag{2.17}$$

where $d_k \in \{0,1,\ldots,b-1\}$, for $k \in I_m$.

We specify that by convention we have $0^0 = 1$. Let the search domains be:

$$\begin{aligned}
Dc_2 &= \{2,3\}\,,\ \text{where } 3 = 3\cdot 1^1\,, && (2.18)\\
Dc_3 &= \{3,4,\ldots,16\}\,,\ \text{where } 16 = 4\cdot 2^2\,, && (2.19)\\
Dc_4 &= \{4,5,\ldots,135\}\,,\ \text{where } 135 = 5\cdot 3^3\,, && (2.20)\\
Dc_5 &= \{5,6,\ldots,1536\}\,,\ \text{where } 1536 = 6\cdot 4^4\,, && (2.21)\\
Dc_6 &= \{6,7,\ldots,21875\}\,,\ \text{where } 21875 = 7\cdot 5^5\,, && (2.22)\\
Dc_7 &= \{7,8,\ldots,373248\}\,,\ \text{where } 373248 = 8\cdot 6^6\,, && (2.23)\\
Dc_8 &= \{8,9,\ldots,7411887\}\,,\ \text{where } 7411887 = 9\cdot 7^7\,, && (2.24)
\end{aligned}$$

and

$$Dc_b = \left\{b, b+1, \ldots, 2\cdot 10^7\right\} \ \text{for}\ \ b = 9,10,\ldots,16\,. \tag{2.25}$$

These search domains were determined by inequality $b^{m-1} \geq m \cdot (b-1)^{b-1}$, where b^{m-1} is the smallest number of m digits in numeration base b, and $m \cdot (b-1)^{b-1}$ is the biggest number that fulfills the condition (2.17) for being a *Münchhaussen number*. The solution, in relation to m, of the equation $\log_b(b^{m-1}) = \log_b(m \cdot (b-1)^{b-1})$ is the number of digits in base b of the number from which we can't have anymore *Münchhaussen number*. The number of digits in base b for *Münchhaussen number* are in Table 2.18.

b	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
n_d	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17

Table 2.18: The number of digits in base b for Münchhaussen number

The limits from where the search for *Münchhaussen numbers* is useless are $n_d(b-1)^{b-1}$, where n_d is taken from the Table 2.18 corresponding to b. The limits are to be found in the search domains (2.18–2.24). For limits bigger than $2 \cdot 10^7$ we considered the limit $2 \cdot 10^7$.

There were determined all *Münchhausen numbers* in base $b = 2, 3, \ldots, 16$ on search domains (2.18–2.25) .

Table 2.19: Münchhausen numbers of (2.18–2.25)

$2 =$	$10_{(2)} =$	$1^1 + 0^0$
$5 =$	$12_{(3)} =$	$1^1 + 2^2$
$8 =$	$22_{(3)} =$	$2^2 + 2^2$
$29 =$	$131_{(4)} =$	$1^1 + 3^3 + 1^1$
$55 =$	$313_{(4)} =$	$3^3 + 1^1 + 3^3$
	—	
$3164 =$	$22352_{(6)} =$	$2^2 + 2^2 + 3^3 + 5^5 + 2^2$
$3416 =$	$23452_{(6)} =$	$2^2 + 3^3 + 4^4 + 5^5 + 2^2$
$3665 =$	$13454_{(7)} =$	$1^1 + 3^3 + 4^4 + 5^5 + 4^4$
	—	
$28 =$	$31_{(9)} =$	$3^3 + 1^1$
$96446 =$	$156262_{(9)} =$	$1^1 + 5^5 + 6^6 + 2^2 + 6^6 + 2^2$
$923362 =$	$1656547_{(9)} =$	$1^1 + 6^6 + 5^5 + 6^6 + 5^5 + 4^4 + 7^7$
$3435 =$	$3435_{(10)} =$	$3^3 + 4^4 + 3^3 + 5^5$
	—	
	—	
$93367 =$	$33661_{(13)} =$	$3^3 + 3^3 + 6^6 + 6^6 + 1^1$

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$31 =$	$23_{(14)} =$	2^2+3^3
$93344 =$	$26036_{(14)} =$	$2^2+6^6+0^0+3^3+6^6$
	—	
	—	

Of course, to these solutions we can also add the trivial solution 1 in any numeration base. If we make the convention $0^0 = 0$ then in base 10 we also have the *Münchhausen number* 438579088, [Sloane, 2014, A046253].

Returning to the convention $0^0 = 1$we can say that the number $\overline{d_1 d_2 \ldots d_m}_{(b)}$ is almost *Münchhausen number* meaning that

$$\left|\overline{d_1 d_2 \ldots d_m}_{(b)} - \left(d_1^{d_1} + d_2^{d_2} + \ldots + d_m^{d_m}\right)\right| \le \varepsilon \ ,$$

where ε can be 0 and then we have *Münchhausen numbers*, or if we have $\varepsilon = 1, 2, \ldots$ then we can say that we have almost *Münchhausen numbers* with the difference of most 1, 2, ... In this regard, the number 438579088 is an almost *Münchhausen number* with the difference of most 1.

2.4.4 Numbers with Digits Sum in Ascending Powers

The numbers which fulfill the condition

$$n_{(b)} = \overline{d_1 d_2 \ldots d_m}_{(b)} = \sum_{k=1}^{m} d_k^k \tag{2.26}$$

are *numbers with digits sum in ascending powers*. In base 10 we have the following *numbers with digits sum in ascending powers* $89 = 8^1+9^2$, $135 = 1^1+3^2+5^3$, $175 = 1^1+7^2+5^3$, $518 = 5^1+1^2+8^3$, $598 = 5^1+9^2+8^3$, $1306 = 1^1+3^2+0^3+6^4$, $1676 = 1^1+6^2+7^3+6^4$, $2427 = 2^1+4^2+2^3+7^4$, [Weisstein, 2014d], [Sloane, 2014, A032799].

We also have the trivial solutions $1, 2, \ldots, b-1$, numbers with property (2.26) for any base $b \ge 2$.

Let $n_{(b)}$ a number in base b with m digits and the equation

$$\log_b(b^{m-1}) = \log_b\left(\frac{b-1}{b-2}[(b-1)^m - 1]\right) \ . \tag{2.27}$$

Lemma 2.11. *The numbers $n_{(b)}$ with property* (2.26) *can't have more than n_d digits, where $n_d = \lceil s \rceil$, and s is the solution of the equation* (2.27).

Proof. The smallest number in base b with m digits is b^{m-1}.The biggest number with property (2.26) is

$$(b-1)^1+(b-1)^2+\ldots+(b-1)^m=(b-1)\frac{(b-1)^m-1}{(b-1)-1}=\frac{b-1}{b-2}[(b-1)^m-1]\ .$$

If the smallest number in base b with m digits is bigger than the bigger number with property (2.26), i.e.

$$b^{m-1}\geq\frac{b-1}{b-2}[(b-1)^m-1]\ ,$$

then the inequality provides the limit of numbers that can fulfill the condition (2.26).

If we logarithmize both terms of the inequation, we obtain an inequation that establishes the limit of possible digits number. It gets us to the solution of the equation (2.27). Let s be the solution of the equation (2.27), but keeping into account that the digits number of a number is an integer, it follows that $n_d=\lceil s\rceil$. □

Corollary 2.12. *The maximum digits number of numbers in base* $b\geq 3$, *which fulfill the condition* (2.26) *are given in the Table* 2.20.

b	3	4	5	6	7	8	9	10	11	12	13	14	15	16
n_d	5	7	9	12	14	17	20	23	27	30	34	37	41	45

Table 2.20: The maximum digits number of numbers in base $b\geq 3$

Let the search domains be defined by

$$Dc_b=\left\{b,b+1,\ldots,\frac{b-1}{b-2}\left[(b-1)^{n_d}-1\right]\right\}\ , \tag{2.28}$$

where $\frac{b-1}{b-2}\left[(b-1)^{n_d}-1\right]$, $n_{(b)}\leq 2\cdot 10^7$, and n_d are given in the Table (2.20). We avoid numbers having only one digit 0, 1, 2, ..., $b-1$, since they are trivial solutions, because $0=0^1$, $1=1^1$, $2=2^1$, ..., $b-1=(b-1)^1$, for any base b.

Then, the search domains are:

$$Dc_3 = \{3,4,\ldots,62\}\ ,\ \text{where } 62=2(2^5-1)\ , \tag{2.29}$$

$$Dc_4 = \{4,5,\ldots,3279\}\ ,\ \text{where } 3279=3\left(3^7-1\right)/2\ , \tag{2.30}$$

$$Dc_5 = \{5,6,\ldots,349524\}\ ,\ \text{where } 349524=4\left(4^9-1\right)/3\ , \tag{2.31}$$

and for $b=6,7,\ldots,16$ the search domains are:

$$Dc_b=\left\{b,b+1,\ldots,2\cdot 10^7\right\}\ . \tag{2.32}$$

All the *numbers with digits sum in ascending powers* in numeration bases $b=2,3,\ldots,16$ on search domains (2.29–2.32) are given in the following table.

Table 2.21: Numbers with the property (2.26) of (2.29–2.32)

$5 =$	$12_{(3)} =$	1^1+2^2
$11 =$	$23_{(4)} =$	2^1+3^2
$83 =$	$1103_{(4)} =$	$1^1+1^2+0^3+3^4$
$91 =$	$1123_{(4)} =$	$1^1+1^2+2^3+3^4$
$19 =$	$34_{(5)} =$	3^1+4^2
$28 =$	$103_{(5)} =$	$1^1+0^2+3^3$
$259 =$	$2014_{(5)} =$	$2^1+0^2+1^3+4^4$
$1114 =$	$13424_{(5)} =$	$1^1+3^2+4^3+2^4+4^5$
$81924 =$	$10110144_{(5)} =$	$1^1+0^2+1^3+1^4+0^5+4^6+4^7$
$29 =$	$45_{(6)} =$	4^1+5^2
$10 =$	$13_{(7)} =$	1^1+3^2
$18 =$	$24_{(7)} =$	2^1+4^2
$41 =$	$56_{(7)} =$	5^1+6^2
$74 =$	$134_{(7)} =$	$1^1+3^2+4^3$
$81 =$	$144_{(7)} =$	$1^1+4^2+4^3$
$382 =$	$1054_{(7)} =$	$1^1+0^2+5^3+4^4$
$1336 =$	$3616_{(7)} =$	$3^1+6^2+1^3+6^4$
$1343 =$	$3626_{(7)} =$	$3^1+6^2+2^3+6^4$
$55 =$	$67_{(8)} =$	6^1+7^2
$8430 =$	$20356_{(8)} =$	$2^1+0^2+3^3+5^4+6^5$
$46806 =$	$133326_{(8)} =$	$1^1+3^2+3^3+3^4+2^5+6^6$
$71 =$	$78_{(9)} =$	7^1+8^2
$4445 =$	$6078_{(9)} =$	$6^1+0^2+7^3+8^4$
$17215 =$	$25547_{(9)} =$	$2^1+5^2+5^3+4^4+7^5$
17621783	$36137478_{(9)} =$	$3^1+6^2+1^3+3^4+7^5+4^6+7^7+8^8$
$89 =$	$89_{(10)} =$	8^1+9^2
$135 =$	$135_{(10)} =$	$1^1+3^2+5^3$
$175 =$	$175_{(10)} =$	$1^1+7^2+5^3$
$518 =$	$518_{(10)} =$	$5^1+1^2+8^3$
$598 =$	$598_{(10)} =$	$5^1+9^2+8^3$
$1306 =$	$1306_{(10)} =$	$1^1+3^2+0^3+6^4$
$1676 =$	$1676_{(10)} =$	$1^1+6^2+7^3+6^4$
$2427 =$	$2427_{(10)} =$	$2^1+4^2+2^3+7^4$
$2646798 =$	$2646798_{(10)} =$	$2^1+6^2+4^3+6^4+7^5+9^6+8^7$
$27 =$	$25_{(11)} =$	2^1+5^2
$39 =$	$36_{(11)} =$	3^1+6^2
$109 =$	$9a_{(11)} =$	9^1+10^2

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$126 =$	$105_{(11)} =$	$1^1+0^2+5^3$
$525 =$	$438_{(11)} =$	$4^1+3^2+8^3$
$580 =$	$488_{(11)} =$	$4^1+8^2+8^3$
$735 =$	$609_{(11)} =$	$6^1+0^2+9^3$
$1033 =$	$85a_{(11)} =$	$8^1+5^2+10^3$
$1044 =$	$86a_{(11)} =$	$8^1+6^2+10^3$
$2746 =$	$2077_{(11)} =$	$2^1+0^2+7^3+7^4$
$59178 =$	$40509_{(11)} =$	$4^1+0^2+5^3+0^4+9^5$
$63501 =$	$43789_{(11)} =$	$4^1+3^2+7^3+8^4+9^5$
$131 =$	$ab_{(12)} =$	10^1+11^2
$17 =$	$14_{(13)} =$	1^1+4^2
$87 =$	$69_{(13)} =$	6^1+9^2
$155 =$	$bc_{(13)} =$	11^1+12^2
$253 =$	$166_{(13)} =$	$1^1+6^2+6^3$
$266 =$	$176_{(13)} =$	$1^1+7^2+6^3$
$345 =$	$207_{(13)} =$	$2^1+0^2+7^3$
$515 =$	$308_{(13)} =$	$3^1+0^2+8^3$
$1754 =$	$a4c_{(13)} =$	$10^1+4^2+12^3$
$1819 =$	$a9c_{(13)} =$	$10^1+9^2+12^3$
$250002 =$	$89a3c_{(13)} =$	$8^1+9^2+10^3+3^4+12^5$
$1000165 =$	$29031a_{(13)} =$	$2^1+9^2+0^3+3^4+1^5+10^6$
$181 =$	$cd_{(14)} =$	12^1+13^2
$11336 =$	$41ba_{(14)} =$	$4^1+1^2+11^3+10^4$
$4844251 =$	$90157d_{(14)} =$	$9^1+0^2+1^3+5^4+7^6+13^7$
$52 =$	$37_{(15)} =$	3^1+7^2
$68 =$	$48_{(15)} =$	4^1+8^2
$209 =$	$de_{(15)} =$	13^1+14^2
$563 =$	$278_{(15)} =$	$2^1+7^2+8^3$
$578 =$	$288_{(15)} =$	$2^1+8^2+8^3$
$15206 =$	$478b_{(15)} =$	$4^1+7^2+8^3+11^4$
$29398 =$	$8a9d_{(15)} =$	$8^1+10^2+9^3+13^4$
$38819 =$	$b77e_{(15)} =$	$11^1+7^2+7^3+14^4$
$38 =$	$26_{(16)} =$	2^1+6^2
$106 =$	$6a_{(16)} =$	6^1+10^2
$239 =$	$ef_{(16)} =$	14^1+15^2
$261804 =$	$3feac_{(16)} =$	$3^1+15^2+14^3+10^4+12^5$

Observation 2.13. T he numbers $\overline{(b-2)(b-1)}_{(b)}$ are *numbers with digits sum in ascending powers* for any base $b \in \mathbb{N}^*$, $b \geq 2$. Indeed, we have the identity

$(b-2)^1+(b-1)^2=(b-2)b+(b-1)$ true for any base $b\in\mathbb{N}^*$, $b\geq 2$, identity which proves the assertion.

2.4.5 Numbers with Digits Sum in Descending Powers

Let the number n with m digits in base b, i.e. $n_{(b)}=\overline{d_1d_2\ldots d_m}_{(b)}$, where $d_k\in\{0,1,\ldots,b-1\}$ for any $k\in I_m$. Let us determine the numbers that fulfill the condition

$$n_{(b)}=d_1\cdot b^{m-1}+d_2\cdot b^{m-2}+\ldots+d_m\cdot b^0=d_1^m+d_2^{m-1}+\ldots+d_m^1\ .$$

Such numbers do not exist because

$$n_{(b)}=d_1\cdot b^{m-1}+d_2\cdot b^{m-2}+\ldots+d_m\cdot b^0>d_1^m+d_2^{m-1}+\ldots+d_m^1\ .$$

Naturally, we will impose the following condition for numbers with digits sum in descending powers:

$$n_{(b)}=d_1\cdot b^{m-1}+d_2\cdot b^{m-2}+\ldots+d_m\cdot b^0=d_1^{m+1}+d_2^m+\ldots+d_m^2\ . \tag{2.33}$$

The biggest number with m digits in base b is b^m-1. The biggest number with m digits sum in descending powers (starting with power $m+1$) is $(b-1)^{m+1}+(b-1)^m+\ldots+(b-1)^2$. If the inequation $(b-1)^2[(b-1)^m-1]/(b-2)\leq b^m-1$, relation to m, has integer solutions ≥ 1, then the condition (2.33) makes sens. The inequation reduces to solving the equation

$$(b-1)^2\frac{(b-1)^m-1}{b-2}=b^m-1\ ,$$

which, after logarithmizing in base b can be solved in relation to m. The solution m represents the digits number of the number $n_{(b)}$, therefore we should take the superior integer part of the solution..

The maximum digits number of the numbers in base $b\geq 3$, which fulfills the condition (2.33) are given Table 2.22.

b	3	4	5	6	7	8	9	10	11	12	13	14	15	16
n_d	7	11	15	20	26	32	38	44	51	58	65	72	79	86

Table 2.22: The maximum digits number of the numbers in base $b\geq 3$

Let the search domain be defined by

$$Dc_b=\left\{b,b+1,\ldots,\frac{(b-1)^2}{b-2}\left[(b-1)^m-1\right]\right\}\ , \tag{2.34}$$

where $(b-1)^2\left[(b-1)^m-1\right]/(b-2)$ is not bigger than $2\cdot 10^7$, and n_d are given in the Table 2.22. We avoid numbers 1, 2, ..., $b-1$, because $0=0^2$ and $1=1^2$ are trivial solutions, and $2\neq 2^2$, $3\neq 3^2$, ..., $b-1\neq (b-1)^2$.

Therefore, the search domains are:

$$Dc_3 = \{3,4,\ldots,508\}\,,\ \text{where } 508=2^2(2^7-1) \tag{2.35}$$
$$Dc_4 = \{4,5,\ldots,797157\}\,,\ \text{where } 797157=3^2(3^{11}-1)/2 \tag{2.36}$$

and for $b=5,6,\ldots,16$ the search domains are

$$Dc_b=\left\{b,b+1,\ldots,2\cdot 10^7\right\}\ . \tag{2.37}$$

All the numbers having the digits sum in descending powers in numeration bases $b=2,3,\ldots,16$ on the search domain (2.35–2.37) are given in the table below.

Table 2.23: Numbers with the property (2.33) of (2.35–2.37)

$5=$	$12_{(3)}=$	1^3+2^2
$20=$	$202_{(3)}=$	$2^4+0^3+2^2$
$24=$	$220_{(3)}=$	$2^4+2^3+0^2$
$25=$	$221_{(3)}=$	$2^4+2^3+1^2$
$8=$	$20_{(4)}=$	2^3+0^2
$9=$	$21_{(4)}=$	2^3+1^2
$28=$	$130_{(4)}=$	$1^4+3^3+0^2$
$29=$	$131_{(4)}=$	$1^4+3^3+1^2$
$819=$	$30303_{(4)}=$	$3^6+0^5+3^4+0^3+3^2$
$827=$	$30323_{(4)}=$	$3^6+0^5+3^4+2^3+3^2$
$983=$	$33113_{(4)}=$	$3^6+3^5+1^4+1^3+3^2$
$12=$	$22_{(5)}=$	2^3+2^2
$44=$	$134_{(5)}=$	$1^4+3^3+4^2$
$65874=$	$4101444_{(5)}=$	$4^8+1^7+0^6+1^5+4^4+4^3+4^2$
$10=$	$13_{(7)}=$	1^3+3^2
$17=$	$23_{(7)}=$	2^3+3^2
$81=$	$144_{(7)}=$	$1^4+4^3+4^2$
$181=$	$346_{(7)}=$	$3^4+4^3+6^2$
$256=$	$400_{(8)}=$	$4^4+0^3+0^2$
$257=$	$401_{(8)}=$	$4^4+0^3+1^2$
$1683844=$	$6330604_{(8)}=$	$6^8+3^7+3^6+0^5+6^4+0^3+4^2$
$1683861=$	$6330625_{(8)}=$	$6^8+3^7+3^6+0^5+6^4+2^3+5^2$
$1685962=$	$6334712_{(8)}=$	$6^8+3^7+3^6+4^5+7^4+1^3+2^2$

Continued on next page

$27=$	$30_{(9)}=$	3^3+0^2
$28=$	$31_{(9)}=$	3^3+1^2
$126=$	$150_{(9)}=$	$1^4+5^3+0^2$
$127=$	$151_{(9)}=$	$1^4+5^3+1^2$
$297=$	$360_{(9)}=$	$3^4+6^3+0^2$
$298=$	$361_{(9)}=$	$3^4+6^3+1^2$
$2805=$	$3756_{(9)}=$	$3^5+7^4+5^3+6^2$
$3525=$	$4746_{(9)}=$	$4^5+7^4+4^3+6^2$
$4118=$	$5575_{(9)}=$	$5^5+5^4+7^3+6^2$
$24=$	$24_{(10)}=$	2^3+4^2
$1676=$	$1676_{(10)}=$	$1^5+6^4+7^3+6^2$
$4975929=$	$4975929_{(10)}=$	$4^8+9^7+7^6+5^5+9^4+2^3+9^2$
$36=$	$33_{(11)}=$	3^3+3^2
$8320=$	$6284_{(11)}=$	$6^5+2^4+8^3+4^2$
$786=$	$556_{(12)}=$	$5^4+5^3+6^2$
$8318=$	$4992_{(12)}=$	$4^5+9^4+9^3+2^2$
$11508=$	$67b0_{(12)}=$	$6^5+7^4+11^3+0^2$
$11509=$	$67b1_{(12)}=$	$6^5+7^4+11^3+1^2$
$17=$	$14_{(13)}=$	1^3+4^2
$43=$	$34_{(13)}=$	3^3+4^2
$253=$	$166_{(13)}=$	$1^4+6^3+6^2$
$784=$	$484_{(13)}=$	$4^4+8^3+4^2$
$33=$	$25_{(14)}=$	2^3+5^2
$1089=$	$57b_{(14)}=$	$5^4+7^3+11^2$
$7386=$	$2998_{(14)}=$	$2^5+9^4+9^3+8^2$
$186307=$	$4bc79_{(14)}=$	$4^6+11^5+12^4+7^3+9^2$
$577=$	$287_{(15)}=$	$2^4+8^3+7^2$
$810=$	$390_{(15)}=$	$3^4+9^3+0^2$
$811=$	$391_{(15)}=$	$3^4+9^3+1^2$
$1404=$	$639_{(15)}=$	$6^4+3^3+9^2$
$16089=$	$4b79_{(15)}=$	$4^5+11^4+7^3+9^2$
$22829=$	$6b6e_{(15)}=$	$6^5+11^4+6^3+14^2$
$64=$	$40_{(16)}=$	4^3+0^2
$65=$	$41_{(16)}=$	4^3+1^2
$351=$	$15f_{(16)}=$	$1^4+5^3+15^2$
$32768=$	$8000_{(16)}=$	$8^5+0^4+0^3+0^2$
$32769=$	$8001_{(16)}=$	$8^5+0^4+0^3+1^2$
$32832=$	$8040_{(16)}=$	$8^5+0^4+4^3+0^2$
$32833=$	$8041_{(16)}=$	$8^5+0^4+4^3+1^2$

Continued on next page

$33119 =$	$815f_{(16)} =$	$8^5+1^4+5^3+15^2$
$631558 =$	$9a306_{(16)} =$	$9^6+10^5+3^4+0^3+6^2$
$631622 =$	$9a346_{(16)} =$	$9^6+10^5+3^4+4^3+6^2$
$631868 =$	$9a43c_{(16)} =$	$9^6+10^5+4^4+3^3+12^2$

2.5 Multifactorial

Function 2.14. By definition, the multifactorial function, [Weisstein, 2014b], is

$$kf(n,k)=\prod_{j=1}^{\lfloor\frac{n}{k}\rfloor}\left(j\cdot k+\mod(n,k)\right). \tag{2.38}$$

For this function, in general bibliography, the commonly used notation are $n!$ for factorial, $n!!$ for double factorial, … .

The well known factorial function is $n!=1\cdot2\cdot3\cdots n$, [Sloane, 2014, A000142]. In general, for 0! the convention $0!=1$ is used.

The double factorial function, [Sloane, 2014, A006882], can also be defined in the following way:

$$n!!=\begin{cases}1\cdot3\cdot5\cdots n, & \text{if} \quad \mod(n,2)=1;\\ 2\cdot4\cdot6\cdots n, & \text{if} \quad \mod(n,2)=0.\end{cases}$$

It is natural to consider the convention $0!!=1$. Let us note that $n!!$ is not the same as $(n!)!$.

The triple factorial function, [Sloane, 2014, A007661], is defined by:

$$n!!!=\begin{cases}1\cdot4\cdot7\cdots n, & \text{if} \quad \mod(n,3)=1;\\ 2\cdot5\cdot8\cdots n, & \text{if} \quad \mod(n,3)=2;\\ 3\cdot6\cdot9\cdots n, & \text{if} \quad \mod(n,3)=0.\end{cases}$$

We will use the same convention for the triple factorial function, $0!!!=1$.

Program 2.15. Program for calculating the multifactorial

```
kf(n,k) := | return 1 if n=0
           | f ← 1
           | r ← mod(n,k)
           | i ← k if r=0
           | i ← r otherwise
           | for j = i, i+k..n
           |   f ← f·j
           | return f
```

This function allows us to calculate $n!$ by the call $kf(n,1)$, $n!!$ by the call $kf(n,2)$, etc.

2.5.1 Factorions

The numbers that fulfill the condition

$$\overline{d_1 d_2 \ldots d_m}_{(b)} = \sum_{k=1}^{n} d_k! \tag{2.39}$$

are called *factorion numbers*, [Gardner, 1978, p. 61 and 64], [Madachy, 1979, 167], [Pickover, 1995, pp. 169–171 and 319–320], [Sloane, 2014, A014080].

Let $n_{(b)}$ a number in base b with m digits and the equation

$$\log_b(b^{m-1}) = \log_b(m(b-1)!) \ . \tag{2.40}$$

Lemma 2.16. *The numbers $n_{(b)}$ with the property* (2.39) *can't have more then n_d digits, where $n_d = \lceil s \rceil$, and s is the solution of the equation* (2.40).

Proof. The smallest number in base b with m digits is b^{m-1}. The biggest number in base b with the property (2.39) is the number $m(b-1)!$. Therefore, the inequality $b^{m-1} \ge m(b-1)!$ provides the limit of numbers that can fulfill the condition (2.39).

If we logarithmize both terms of inequation $b^{m-1} \ge m(b-1)!$ we get an inequation that establishes the limit of possible digits number for the numbers that fulfill the condition (2.39). It drives us to the solution of the equation (2.40). Let s be the solution of the equation (2.40), but keeping into account that the digits number of a number is an integer, it follows that $n_d = \lceil s \rceil$. □

Corollary 2.17. *The maximum digits number of the numbers in base b, which fulfills the condition* (2.39) *are given below.*

b	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
n_d	2	3	4	4	5	6	6	7	8	9	9	10	11	12	12

Table 2.24: The maximum digits number of the numbers in base b

Let the search domains be defined by

$$Dc_b = \{b, b+1, \ldots, n_d(b-1)!\} \ , \tag{2.41}$$

where $n_d(b-1)!$ is not bigger than $2 \cdot 10^7$, and n_d are values from the Table 2.24.

Threfore, the search domains are:

$$\begin{aligned} Dc_2 &= \{2\}\ , & (2.42)\\ Dc_3 &= \{3,4,\ldots,6\}\ , & (2.43)\\ Dc_4 &= \{4,5,\ldots,24\}\ , & (2.44)\\ Dc_5 &= \{5,6,\ldots,96\}\ , & (2.45)\\ Dc_6 &= \{6,7,\ldots,600\}\ , & (2.46)\\ Dc_7 &= \{7,8,\ldots,4320\}\ , & (2.47)\\ Dc_8 &= \{8,9,\ldots,30240\}\ , & (2.48)\\ Dc_9 &= \{9,10,\ldots,282240\}\ , & (2.49)\\ Dc_{10} &= \{10,11,\ldots,2903040\}\ , & (2.50) \end{aligned}$$

and for $b=10,11,\ldots,16$,

$$Dc_b = \left\{b, b+1, \ldots, 2\cdot 10^7\right\}\ . \qquad (2.51)$$

In base 10 we have only 4 factorions $1=1!$, $2=2!$, $145=1!+4!+5!$ and $40585=4!+0!+5!+8!+5!$, with the observation that by convention we have $0!=1$. To note that we have the trivial solutions $1=1!$ and $2=2!$ in any base of numeration $b\ge 3$, and $3!\ne 3$, $\ldots$, $(b-1)!\ne b-1$ for any $b\ge 3$. The list of factorions in numeration base $b=2,3,\ldots,16$ on the search domain given by (2.42–2.51) is:

Table 2.25: Numbers with the property (2.39) of (2.42–2.51)

$2=$	$10_{(2)}=$	$1!+0!$
—		
$7=$	$13_{(4)}=$	$1!+3!$
$49=$	$144_{(5)}=$	$1!+4!+4!$
$25=$	$41_{(6)}=$	$4!+1!$
$26=$	$42_{(6)}=$	$4!+2!$
—		
—		
$41282=$	$62558_{(9)}=$	$6!+2!+5!+5!+8!$
$145=$	$145_{(10)}=$	$1!+4!+5!$
$40585=$	$40585_{(10)}=$	$4!+0!+5!+8!+5!$
$26=$	$24_{(11)}=$	$2!+4!$
$48=$	$44_{(11)}=$	$4!+4!$
$40472=$	$28453_{(11)}=$	$2!+8!+4!+5!+3!$
—		

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—		
—		
$1441 =$	$661_{(15)} =$	$6!+6!+1!$
$1442 =$	$662_{(15)} =$	$6!+6!+2!$
—		

2.5.2 Double Factorions

We can also define the *duble factorion* numbers, i.e. the numbers which fulfills the condition

$$\overline{d_1 d_2 \ldots d_m}_{(b)} = \sum_{k=1}^{m} d_k!! \,. \tag{2.52}$$

Let $n_{(b)}$ be a number in base b with m digits and the equation

$$\log_b(b^{m-1}) = \log_b(m \cdot b!!) \ . \tag{2.53}$$

Lemma 2.18. *The numbers $n_{(b)}$ with the property* (2.52) *can't have more than n_d digits, where $n_d = \lceil s \rceil$, and s is the solution of the equation* (2.53).

Proof. The smallest number in base b with m digits is b^{m-1} . The biggest number in base b with the property (2.52) is the number $m \cdot b!!$. Therefore, the inequality $b^{m-1} \geq m \cdot b!!$ provides a limit of numbers that can fulfill the condition (2.52).

If we logarithmize both terms of inequation $b^{m-1} \geq m \cdot b!!$ we get an inequation which establishes the limit of possible digits number for the numbers which fulfills the condition (2.52).It drives us to the solution of the equation (2.53). Let s the solution of the equation (2.53), but keeping into account that the digits number of a number is an integer, it follows that $n_d = \lceil s \rceil$. □

Corollary 2.19. *The maximum digits of numbers in base b, which fulfills the condition* (2.52) *are given in the table below.*

b	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
n_d	1	2	2	3	3	3	4	4	4	5	5	6	6	6	7

Table 2.26: The maximum digits of numbers in base b

Let the search domains be defined by

$$Dc_b = \{b, b+1, \dots, n_d(b-1)!!\} \ , \tag{2.54}$$

where n_d are values from the Table (2.26).

Therefore, the search domains are:

$$\begin{aligned}
Dc_2 &= \{2\}\,, &(2.55)\\
Dc_3 &= \{3,4\}\,, &(2.56)\\
Dc_4 &= \{4,5,6\}\,, &(2.57)\\
Dc_5 &= \{5,6,\dots,24\}\,, &(2.58)\\
Dc_6 &= \{6,7,\dots,45\}\,, &(2.59)\\
Dc_7 &= \{7,8,\dots,144\}\,, &(2.60)\\
Dc_8 &= \{8,9,\dots,420\}\,, &(2.61)\\
Dc_9 &= \{9,10,\dots,1536\}\,, &(2.62)\\
Dc_{10} &= \{10,11,\dots,3780\}\,, &(2.63)\\
Dc_{11} &= \{11,12,\dots,19200\}\,, &(2.64)\\
Dc_{12} &= \{12,13,\dots,51975\}\,, &(2.65)\\
Dc_{13} &= \{13,14,\dots,276480\}\,, &(2.66)\\
Dc_{14} &= \{14,15,\dots,810810\}\,, &(2.67)\\
Dc_{15} &= \{15,16,\dots,3870720\}\,, &(2.68)\\
Dc_{16} &= \{16,17,\dots,14189175\}\,, &(2.69)
\end{aligned}$$

In the Table 2.27 we have all the double factorions for numeration bases $b = 2,3,\dots,16$.

Table 2.27: Numbers with the property (2.52) of (2.55–2.69)

$2 =$	$10_{(2)} =$	$1!! + 0!!$
$-$		
$-$		
$9 =$	$14_{(5)} =$	$1!! + 4!!$
$17 =$	$25_{(6)} =$	$2!! + 5!!$
$97 =$	$166_{(7)} =$	$1!! + 6!! + 6!!$
$49 =$ $50 =$ $51 =$	$61_{(8)} =$ $62_{(8)} =$ $63_{(8)} =$	$6!! + 1!!$ $6!! + 2!!$ $6!! + 3!!$

Continued on next page

$400 =$	$484_{(9)} =$	$4!! + 8!! + 4!!$
$107 =$	$107_{(10)} =$	$1!! + 0!! + 7!!$
$16 =$	$15_{(11)} =$	$1!! + 5!!$
$1053 =$	$739_{(12)} =$	$7!! + 3!! + 9!!$
$-$		
$1891 =$	$991_{(14)} =$	$9!! + 9!! + 1!!$
$1892 =$	$992_{(14)} =$	$9!! + 9!! + 2!!$
$1893 =$	$993_{(14)} =$	$9!! + 9!! + 3!!$
$191666 =$	$4dbc6_{(14)} =$	$4!! + 13!! + 11!! + 12!! + 6!!$
$51 =$	$36_{(15)} =$	$3!! + 6!!$
$96 =$	$66_{(15)} =$	$6!! + 6!!$
$106 =$	$71_{(15)} =$	$7!! + 1!!$
$107 =$	$72_{(15)} =$	$7!! + 2!!$
$108 =$	$73_{(15)} =$	$7!! + 3!!$
$181603 =$	$38c1d_{(15)} =$	$3!! + 8!! + 12!! + 1!! + 13!!$
$2083607 =$	$1fcb17_{(16)} =$	$1!! + 15!! + 12!! + 11!! + 1!! + 7!!$

2.5.3 Triple Factorials

The *triple factorion* numbers are the numbers fulfilling the condition

$$\overline{d_1 d_2 \dots d_m}_{(b)} = \sum_{k=1}^{m} d_k!!! \,. \tag{2.70}$$

Let $n_{(b)}$ a number in base b with m and the equation

$$\log_b(b^{m-1}) = \log_b(m \cdot b!!!) \ . \tag{2.71}$$

Lemma 2.20. *The numbers $n_{(b)}$ with property* (2.70) *can't have more n_d digits, where $n_d = \lceil s \rceil$, and s is the solution of the equation* (2.71).

Proof. The smallest number in base b with m digits is b^{m-1} . The biggest number in base b with property (2.70) is the number $m \cdot b!!!$. Therefore, the inequality $b^{m-1} \geq m \cdot b!!!$ provides a limit of numbers that can fulfill the condition(2.70).

If we logarithmize both terms of inequation $b^{m-1} \geq m \cdot b!!!$ we get an inequation that establishes the limit of possible digits number of the numbers that fulfill the condition (2.70). It drives us to the solution of the equation (2.71). Let s be the solution of the equation (2.71), but keeping into account that the digits number of a number is an integer, it follows that $n_d = \lceil s \rceil$. □

b	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
n_d	2	3	3	3	3	4	4	4	4	4	5	5	5	6	6

Table 2.28: The maximum numbers of numbers in base b

Corollary 2.21. *The maximum numbers of numbers, in base b, which fulfills the condition* (2.70) *are given below.*

Let the search domains be defined by

$$Dc_b = \{b, b+1, \ldots, n_d(b-1)!!!\}\ , \tag{2.72}$$

where n_d are the values in the Table 2.28.

Therefore, the search domains are:

$$\begin{aligned}
Dc_2 &= \{2\}\,, && (2.73)\\
Dc_3 &= \{3,4,5,6\}\,, && (2.74)\\
Dc_4 &= \{4,5,\ldots,9\}\,, && (2.75)\\
Dc_5 &= \{5,6,\ldots,12\}\,, && (2.76)\\
Dc_6 &= \{6,7,\ldots,30\}\,, && (2.77)\\
Dc_7 &= \{7,8,\ldots,72\}\,, && (2.78)\\
Dc_8 &= \{8,9,\ldots,112\}\,, && (2.79)\\
Dc_9 &= \{9,10,\ldots,320\}\,, && (2.80)\\
Dc_{10} &= \{10,11,\ldots,648\}\,, && (2.81)\\
Dc_{11} &= \{11,12,\ldots,1120\}\,, && (2.82)\\
Dc_{12} &= \{12,13,\ldots,4400\}\,, && (2.83)\\
Dc_{13} &= \{13,14,\ldots,9720\}\,, && (2.84)\\
Dc_{14} &= \{14,15,\ldots,18200\}\,, && (2.85)\\
Dc_{15} &= \{15,16,\ldots,73920\}\,, && (2.86)\\
Dc_{16} &= \{16,17,\ldots,174960\}\,. && (2.87)
\end{aligned}$$

In Table 2.29 we have all the triple factorions for numeration bases $b = 2,3,\ldots,16$.

Table 2.29: Numbers with the property (2.70) of (2.72)

$2 =$	$10_{(2)} =$	$1!!! + 0!!!$
−		

Continued on next page

–		
–		
$11 =$	$15_{(6)} =$	$1!!! + 5!!!$
$20 =$	$26_{(7)} =$	$2!!! + 6!!!$
$31 =$	$37_{(8)} =$	$3!!! + 7!!!$
$161 =$	$188_{(9)} =$	$1!!! + 8!!! + 8!!!$
$81 =$	$81_{(10)} =$	$8!!! + 1!!!$
$82 =$	$82_{(10)} =$	$8!!! + 2!!!$
$83 =$	$83_{(10)} =$	$8!!! + 3!!!$
$84 =$	$84_{(10)} =$	$8!!! + 4!!!$
$285 =$	$23a_{(11)} =$	$2!!! + 3!!! + 10!!!$
–		
$19 =$	$16_{(13)} =$	$1!!! + 6!!!$
–		
$98 =$	$68_{(15)} =$	$6!!! + 8!!!$
$1046 =$	$49b_{(15)} =$	$4!!! + 9!!! + 11!!!$
$3804 =$	$11d9_{(15)} =$	$1!!! + 1!!! + 13!!! + 9!!!$
$282 =$	$11a_{(16)} =$	$1!!! + 1!!! + 10!!!$
$1990 =$	$7c6_{(16)} =$	$7!!! + 12!!! + 6!!!$
$15981 =$	$3e6d_{(16)} =$	$3!!! + 14!!! + 6!!! + 13!!!$

Similarly, one can obtain *quadruple factorions* and *quintuple factorions*. In numeration base 10 we only have factorions $49 = 4!!!! + 9!!!!$ and $39 = 3!!!!! + 9!!!!!$.

2.5.4 Factorial Primes

An important class of numbers that are prime numbers are the *factorial primes*.

Definition 2.22. Numbers of the form $n! \pm 1$ are called *factorial primes*.

In Table 2.30 we have all the *factorial primes*, for $n \le 30$, which are primes.

Table 2.30: Factorial primes that are primes

$1! + 1$	$=$	2
$2! + 1$	$=$	3
$3! - 1$	$=$	5

Continued on next page

$3!+1$	=	7
$4!-1$	=	23
$6!-1$	=	719
$7!-1$	=	5039
$11!+1$	=	39916801
$12!-1$	=	479001599
$14!-1$	=	8717821199
$27!+1$	=	10888869450418352160768000001
$30!-1$	=	265252859812191058636308479999999

Similarly, we can define *double factorial primes*.

Definition 2.23. The numbers of the form $n!! \pm 1$ are called *double factorial primes*.

In Table 2.31 we have all the numbers that are *double factorial primes*, for $n \leq 30$, that are primes.

Table 2.31: Double factorial primes that are primes

$3!!-1$	=	2
$2!!+1$	=	3
$4!!-1$	=	7
$6!!-1$	=	47
$8!!-1$	=	383
$16!!-1$	=	10321919
$26!!-1$	=	51011754393599

Definition 2.24. The numbers of the form $n!!! \pm 1$ are called *triple factorial primes*.

In Table 2.32 we have all the numbers that are *triple factorial primes*, foru $n \leq 30$, that are primes.

Table 2.32: Triple factorial primes that are primes

$3!!!-1$	$=$	2
$4!!!-1$	$=$	3
$4!!!+1$	$=$	5
$5!!!+1$	$=$	11
$6!!!-1$	$=$	17
$6!!!+1$	$=$	19
$7!!!+1$	$=$	29
$8!!!-1$	$=$	79
$9!!!+1$	$=$	163
$10!!!+1$	$=$	281
$11!!!+1$	$=$	881
$17!!!+1$	$=$	209441
$20!!!-1$	$=$	4188799
$24!!!+1$	$=$	264539521
$26!!!-1$	$=$	2504902399
$29!!!+1$	$=$	72642169601

Definition 2.25. The numbers of the form $n!!!!\pm 1$ are called *quadruple factorial primes.*

In Table 2.33 we have all the numbers that are *quadruple factorial primes*,for $n \leq 30$, that are primes.

Table 2.33: Quadruple factorial primes that are primes

$4!!!!-1$	$=$	3
$4!!!!+1$	$=$	5
$6!!!!-1$	$=$	11
$6!!!!+1$	$=$	13
$8!!!!-1$	$=$	31
$9!!!!+1$	$=$	163
$12!!!!-1$	$=$	383
$16!!!!-1$	$=$	6143
$18!!!!+1$	$=$	30241
$22!!!!-1$	$=$	665279
$24!!!!-1$	$=$	2949119

Definition 2.26. The numbers of the form $n!!!!!\pm 1$ are called *quintuple factorial primes.*

In Table 2.34 we have all the numbers that are *quintuple factorial primes*, for $n \leq 30$, that are primes.

Table 2.34: Quintuple factorial primes that are primes

$6!!!!!-1$	=	5
$6!!!!!+1$	=	7
$7!!!!!-1$	=	13
$8!!!!!-1$	=	23
$9!!!!!+1$	=	37
$11!!!!!+1$	=	67
$12!!!!!-1$	=	167
$13!!!!!-1$	=	311
$13!!!!!+1$	=	313
$14!!!!!-1$	=	503
$15!!!!!+1$	=	751
$17!!!!!-1$	=	8857
$23!!!!!+1$	=	129169
$26!!!!!+1$	=	576577
$27!!!!!-1$	=	1696463
$28!!!!!-1$	=	3616703

Definition 2.27. The numbers of the form $n!!!!!!\pm 1$ are called *sextuple factorial primes.*

In Table 2.35 we have all the numbers that are *sextuple factorial primes*, for $n \leq 30$, that are primes.

Table 2.35: Sextuple factorial primes that are primes

$6!!!!!!-1$	=	5
$6!!!!!!+1$	=	7
$8!!!!!!+1$	=	17
$10!!!!!!+1$	=	41

Continued on next page

$12!!!!!!-1$	$=$	71
$12!!!!!!+1$	$=$	73
$14!!!!!!-1$	$=$	223
$16!!!!!!+1$	$=$	641
$18!!!!!!+1$	$=$	1297
$20!!!!!!+1$	$=$	4481
$22!!!!!!+1$	$=$	14081
$28!!!!!!+1$	$=$	394241

2.6 Digital Product

We consider the function dp product of the digits' number $n_{(b)}$.

Program 2.28. The function dp is given of program

$$
dp(n,b):=\left|\begin{array}{l}
v\leftarrow dn(n,b)\\
p\leftarrow 1\\
for\;\; j\in ORIGIN..last(v)\\
\quad p\leftarrow p\cdot v_j\\
return\;\; p
\end{array}\right.
$$

The program 2.28 calls the program 2.2.

Examples:

1. The call $dp(76,8)=4$ verifies with the identity $76_{(10)}=114_{(8)}$ and by the fact than $1\cdot 1\cdot 4=4$;
2. The call $dp(1234,16)=104$ verifies with the identity $1234_{(10)}=4d2_{(16)}$ and by the fact than $4\cdot d\cdot 2=4\cdot 13\cdot 2=104$;
3. The call $dp(15,2)=1$ verifies with the identity $15_{(10)}=1111_{(2)}$ and by the fact than $1\cdot 1\cdot 1\cdot 1=1$.

We suggest to resolve the Diophantine equations

$$
\alpha\cdot\left(d_1^k+d_2^k+\ldots+d_m^k\right)+\beta\cdot(d_1\cdot d_2\cdots d_m)=\overline{d_1d_2\ldots d_m}_{(b)}\ , \tag{2.88}
$$

where $d_1,d_2,\ldots d_m\in\{0,1,\ldots,b-1\}$, iar $\alpha,\beta\in\mathbb{N}$, $b\in\mathbb{N}^*$, $b\geq 2$.

Program 2.29. Program for determining the natural numbers which verifies the equation (2.88):

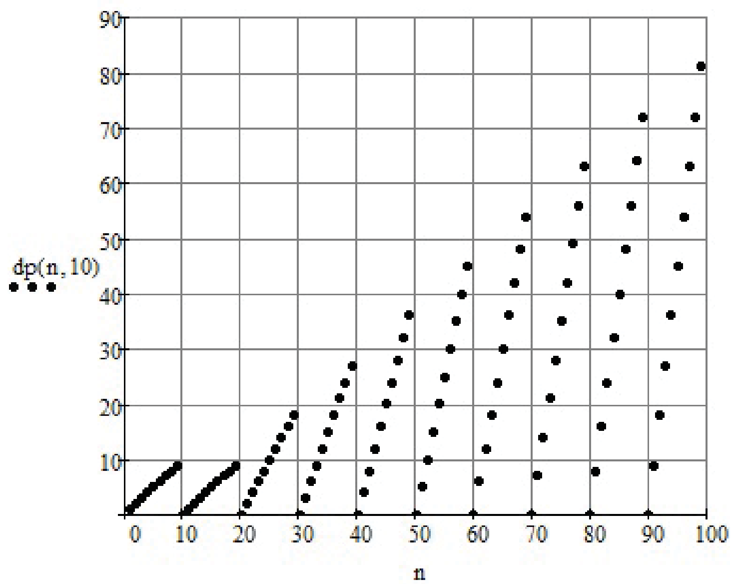

Figure 2.4: The digital product function

$$ED(L,b,k,\alpha,\beta) := \left| \begin{array}{l} S \leftarrow (\text{"}n_{(10)}\text{"} \quad \text{"}n_{(b)}\text{"} \quad \text{"}s\text{"} \quad \text{"}p\text{"} \quad \text{"}\alpha\cdot s\text{"} \quad \text{"}\beta\cdot p\text{"}) \\ for\ n \in b..L \\ \quad \left| \begin{array}{l} s \leftarrow dsk(n,b,k) \\ p \leftarrow dp(n,b) \\ if\ \alpha\cdot s+\beta\cdot p{=}n \\ \quad \left| \begin{array}{l} v \leftarrow (n \quad dn(n,b)^{\mathrm{T}} \quad s \quad p \quad \alpha\cdot s \quad \beta\cdot p) \\ S \leftarrow stack[S,v] \end{array} \right. \end{array} \right. \\ return\ \ S \end{array} \right.$$

Example 2.30.

1. The case $b = 10$, $\alpha = 2$, $\beta = 1$ and $k = 1$ with $n \leq 10^3$ has the solutions:

   (a) $14 = 2\cdot(1+4)+1\cdot(1\cdot4)$;

   (b) $36 = 2\cdot(3+6)+1\cdot(3\cdot6)$;

   (c) $77 = 2\cdot(7+7)+1\cdot(7\cdot7)$.

2. The case $b = 10$, $\alpha = 2$, $\beta = 1$ and $k = 3$ with $n \leq 10^3$ has the solutions:

   (a) $624 = 2\cdot(6^3+2^3+4^3)+1\cdot(6\cdot2\cdot4)$;

   (b) $702 = 2\cdot(7^3+0^3+2^3)+1\cdot(7\cdot0\cdot2)$.

3. The case $b = 11$, $\alpha = 2$, $\beta = 1$ and $k = 3$ with $n \le 10^3$ has the solutions:
   (a) $136_{(10)} = 114_{(11)} = 2 \cdot (1^3 + 1^3 + 4^3) + 1 \cdot (1 \cdot 1 \cdot 4)$.
4. The case $b = 15$, $\alpha = 2$, $\beta = 1$ and $k = 3$ with $n \le 10^3$ has the solutions:
   (a) $952_{(10)} = 437_{(15)} = 2 \cdot (4^3 + 3^3 + 7^3) + 1 \cdot (4 \cdot 3 \cdot)$.
5. The case $b = 10$, $\alpha = 1$, $\beta = 0$ and $k = 3$ with $n \le 10^3$ has the solutions:
   (a) $153_{(10)} = 1^3 + 5^3 + 3^3$;
   (b) $370_{(10)} = 3^3 + 7^3 + 0^3$;
   (c) $371_{(10)} = 3^3 + 7^3 + 1^3$;
   (d) $407_{(10)} = 4^3 + 0^3 + 7^3$.

   These are the Narcissistic numbers in base $b = 10$ of 3 digits, see Table 2.8.

2.7 Sum–Product

Definition 2.31. [Weisstein, 2014e],[Sloane, 2014, A038369] The natural numbers n, in the base b, $n_{(b)} = \overline{d_1 d_2 \ldots d_m}$, where $d_k \in \{0, 1, \ldots, b-1\}$ which verifies the equality

$$n = \prod_{k=1}^{m} d_k \sum_{k=1}^{m} d_k \ , \tag{2.89}$$

are called *sum–product numbers*.

Let the function sp, defined on $\mathbb{N}^* \times \mathbb{N}_{\ge 2}$ with values on $\mathbb{N}^*$:

$$sp(n,b) := dp(n,b) \cdot dks(n,b,1) \ , \tag{2.90}$$

where the digital product function, *dp*, is given by 2.28 and the digital sum–product function of power k, dks, is defined in relation (2.1). The graphic of the function is shown in 2.5.

Program 2.32. for determining the *sum–product* numbers in base b.

$$Psp(L,b,\varepsilon) := \left| \begin{array}{l} j \leftarrow 1 \\ for\ \ n \in 1..L \\ \quad if\ \left|sp(n,b) - n\right| \le \varepsilon \\ \quad \left| \begin{array}{l} q_{j,1} \leftarrow n \\ q_{j,2} \leftarrow dn(n,b)^{\mathrm{T}} \end{array} \right. \\ return\ \ q \end{array} \right.$$

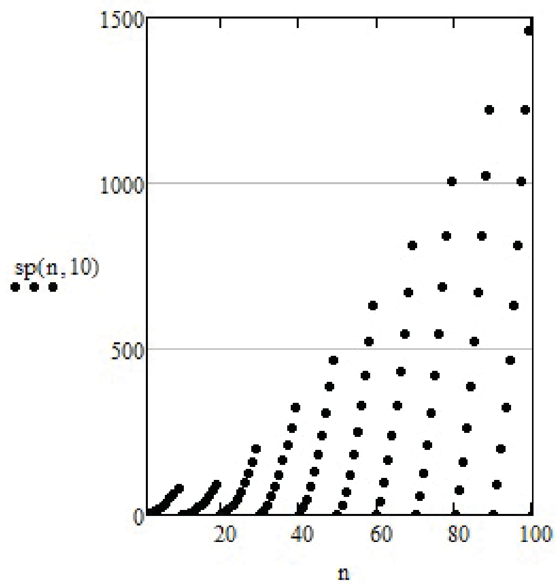

Figure 2.5: Function sp

In the table below we show all *sum–product* numbers in bases $b = 2, 3, \ldots, 16$, up to the limit $L = 10^6$. It is obvious that for any bases $b \geq 2$ the number $n_{(b)} = 1$ is a *sum–product* number, therefore we do not show this trivial solution. The call of program 2.65 is made by command $Psp(10^6, 7, 0) =$.

Table 2.36: Sum–product numbers

$6 = 12_{(4)} = (1+2) \cdot 1 \cdot 2$
$96 = 341_{(5)} = (3+4+1) \cdot 3 \cdot 4 \cdot 1$
$16 = 22_{(7)} = (2+2) \cdot 2 \cdot 2$
$128 = 242_{(7)} = (2+4+2) \cdot 2 \cdot 4 \cdot 2$
$480 = 1254_{(7)} = (1+2+5+4) \cdot 1 \cdot 2 \cdot 5 \cdot 4$
$864 = 2343_{(7)} = (2+3+4+3) \cdot 2 \cdot 3 \cdot 4 \cdot 3$
$21600 = 116655_{(7)} = (1+1+6+6+5+5) \cdot 1 \cdot 1 \cdot 6 \cdot 6 \cdot 5 \cdot 5$
$62208 = 346236_{(7)} = (3+4+6+2+3+6) \cdot 3 \cdot 4 \cdot 6 \cdot 2 \cdot 3 \cdot 6$
$73728 = 424644_{(7)} = (4+2+4+6+4+4) \cdot 4 \cdot 2 \cdot 4 \cdot 6 \cdot 4 \cdot 4$
$12 = 13_{(9)} = (1+3) \cdot 1 \cdot 9$
$172032 = 281876_{(9)} = (2+8+1+8+7+6) \cdot 2 \cdot 8 \cdot 1 \cdot 8 \cdot 7 \cdot 6$
$430080 = 724856_{(9)} = (7+2+4+8+5+6) \cdot 7 \cdot 2 \cdot 4 \cdot 8 \cdot 5 \cdot 6$
$135 = 135_{(10)} = (1+3+5) \cdot 1 \cdot 3 \cdot 5$

Continued on next page

$144 = 144_{(10)} = (1+4+4)\cdot 1\cdot 4\cdot 4$
$300 = 253_{(11)} = (2+5+3)\cdot 2\cdot 5\cdot 3$ $504 = 419_{(11)} = (4+1+9)\cdot 4\cdot 1\cdot 9$ $2880 = 2189_{(11)} = (2+1+8+9)\cdot 2\cdot 1\cdot 8\cdot 9$ $10080 = 7634_{(11)} = (7+6+3+4)\cdot 7\cdot 6\cdot 3\cdot 4$ $120960 = 82974_{(11)} = (8+2+9+7+4)\cdot 8\cdot 2\cdot 9\cdot 7\cdot 4$
$176 = 128_{(12)} = (1+2+8)\cdot 1\cdot 2\cdot 8$ $231 = 173_{(12)} = (1+7+3)\cdot 1\cdot 7\cdot 3$ $495 = 353_{(12)} = (3+5+3)\cdot 3\cdot 5\cdot 3$
$720 = 435_{(13)} = (4+3+5)\cdot 4\cdot 3\cdot 5$ $23040 = a644_{(13)} = (10+6+4+4)\cdot 10\cdot 6\cdot 4\cdot 4$ $933120 = 268956_{(13)} = (2+6+8+9+5+6)\cdot 2\cdot 6\cdot 8\cdot 9\cdot 5\cdot 6$
$624 = 328_{(14)} = (3+2+8)\cdot 3\cdot 2\cdot 8$ $1040 = 544_{(14)} = (5+4+4)\cdot 5\cdot 4\cdot 4$ $22272 = 818c_{(14)} = (8+1+8+12)\cdot 8\cdot 1\cdot 8\cdot 12$
$8000 = 2585_{(15)} = (2+5+8+5)\cdot 2\cdot 5\cdot 8\cdot 5$
$20 = 14_{(16)} = (1+4)\cdot 1\cdot 4$

The program 2.32 *Psp* has 3 input parameters. If the parameter ε is 0 then we will obtain *sum–product* numbers, if $\varepsilon = 1$ then we will obtain *sum–product* numbers and *almost sum–product* numbers. For example, in base $b = 7$ we have the *almost sum–product* numbers:

$$43 = 61_{(7)} = (6+1)\cdot 6 = 42\ ,$$

$$3671 = 13463_{(7)} = (1+3+4+6+3)\cdot 3\cdot 4\cdot 6\cdot 3 = 3672\ ,$$

$$5473 = 21646_{(7)} = (2+1+6+4+6)\cdot 2\cdot 1\cdot 6\cdot 4\cdot 6 = 5472\ ,$$

$$10945 = 43624_{(7)} = (4+3+6+2+4)\cdot 4\cdot 3\cdot 6\cdot 2\cdot 4 = 10944\ .$$

It is clear that the *sum–product* numbers can not be prime numbers. In base 10, up to the limit $L = 10^6$, there is only one *almost sum–product* number which is a prime, that is 13. Maybe there are other *almost sum–product* numbers that are primes?

The number 144 has the quality of being a *sum–product* number and a perfect square. This number is also called "*gross number*" (rough number). There is also another *sum–product* number that is perfect square? At least between numbers displayed in Table 2.36 there is no perfect square excepting 144.

2.8 Code Puzzle

Using the following letter-to-number code:

A	B	C	D	E	F	G	H	I	J	K	L	M
01	05	06	07	08	09	10	11	12	13	14	15	16

N	O	P	Q	R	S	T	U	V	W	X	Y	Z
14	15	16	17	18	19	20	21	22	23	24	25	26

then $c_p(n)$ = the numerical code for the spelling of n in English language; for example: $c_p(ONE) = 151405$, $c_p(TWO) = 202315$, etc.

2.9 Pierced Chain

Let the function

$$c(n) = 101 \cdot 1\underbrace{0001}_{1}\underbrace{0001}_{2}\ldots\underbrace{0001}_{n-1} ,$$

then $c(1) = 101$, $c(10001) = 1010101$, $c(100010001) = 10101010101, \ldots$.

How many $c(n)/101$ are primes? [Smarandache, 2014, 1979, 1993a, 2006].

2.10 Divisor Product

Let $P_d(n)$ is the product of all positive divisors of n.

$$\begin{aligned}
P_d(1) &= 1 = 1 , \\
P_d(2) &= 1 \cdot 2 = 2 , \\
P_d(3) &= 1 \cdot 3 = 3 , \\
P_d(4) &= 1 \cdot 2 \cdot 4 = 8 , \\
P_d(5) &= 1 \cdot 5 = 5 , \\
P_d(6) &= 1 \cdot 2 \cdot 3 \cdot 6 = 36 , \\
&\vdots
\end{aligned}$$

thus, the sequence: 1, 2, 3, 8, 5, 36, 7, 64, 27, 100, 11, 1728, 13, 196, 225, 1024, 17, 5832, 19, 8000, 441, 484, 23, 331776, 125, 676, 729, 21952, 29, 810000, 31, 32768, 1089, 1156, 1225, 100776, 96, 37, 1444, 1521, 2560000, 41, ...

2.11 Proper Divisor Products

Let $P_{dp}(n)$ is the product of all positive proper divisors of n.

$$\begin{aligned}
P_{dp}(1) &= 1\,, \\
P_{dp}(2) &= 1\,, \\
P_{dp}(3) &= 1\,, \\
P_{dp}(4) &= 2\,, \\
P_{dp}(5) &= 1\,, \\
P_{dp}(6) &= 2\cdot 3 = 6\,, \\
&\vdots
\end{aligned}$$

thus, the sequence: 1, 1, 1, 2, 1, 6, 1, 8, 3, 10, 1, 144, 1, 14, 15, 64, 1, 324, 1, 400, 21, 22, 1, 13824, 5, 26, 27, 784, 1, 27000, 1, 1024, 33, 34, 35, 279936, 1, 38, 39, 64000, 1, … .

2.12 n – Multiple Power Free Sieve

Definition 2.33. The sequence of positive integer numbers $\{2, 3, \ldots, L\}$ from which take off numbers $k \cdot p^n$, where $p \in \mathbb{P}_{\geq 2}$, $n \in \mathbb{N}^*$, $n \geq 3$ and $k \in \mathbb{N}^*$ such that $k \cdot p^n \leq L$ (take off all multiples of all n – power primes) is called *n – power free sieve*.

The list of numbers without primes to multiple cubes up to $L = 125$ is: 2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 13, 14, 15, 17, 18, 19, 20, 21, 22, 23, 25, 26, 28, 29, 30, 31, 33, 34, 35, 36, 37, 38, 39, 41, 42, 43, 44, 45, 46, 47, 49, 50, 51, 52, 53, 55, 57, 58, 59, 60, 61, 62, 63, 65, 66, 67, 68, 69, 70, 71, 73, 74, 75, 76, 77, 78, 79, 82, 83, 84, 85, 86, 87, 89, 90, 91, 92, 93, 94, 95, 97, 98, 99, 100, 101, 102, 103, 105, 106, 107, 109, 110, 111, 113, 114, 115, 116, 117, 118, 119, 121, 122, 123, 124 . We eliminated the numbers: 8, 16, 24, 32, 40, 48, 56, 64, 72, 80, 88, 96, 104, 112, 120 (multiples of 2^3), 27, 54, 81, 108 (multiples of 3^3), 125 (multiples of 5^3) .

The list of numbers without multiples of order 4 powers of primes to $L = 125$ is: 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 97, 98, 99, 100, 101, 102, 103, 104, 105, 106, 107, 108, 109, 110, 111, 113, 114, 115, 116, 117, 118, 119, 120, 121, 122, 123, 124, 125 . We eliminated the numbers: 16, 32, 48, 64, 80, 96, 112 (multiples of 2^4) and 81 (multiples of 3^4) .

2.13 Irrational Root Sieve

Definition 2.34. [Smarandache, 2014, 1993a, 2006] The sequence of positive integer numbers $\{2,3,\ldots,L\}$ from which we take off numbers $j \cdot k^2$, where $k = 2,3,\ldots,\lfloor L \rfloor$ and $j = 1,2,\ldots,\lfloor L/k^2 \rfloor$ is the *free sequence of multiples perfect squares.*

The list of numbers free of perfect squares multiples for $L = 71$ is: 2, 3, 5, 6, 7, 10, 11, 13, 14, 15, 17, 19, 21, 22, 23, 26, 29, 30, 31, 33, 34, 35, 37, 38, 39, 41, 42, 43, 46, 47, 51, 53, 55, 57, 58, 59, 61, 62, 65, 66, 67, 69, 70, 71 .

The number of numbers free of perfect squares multiples to the limit L is given in the Table 2.37

Table 2.37: The length of the free of perfect squares multiples

L	*length*
10	6
100	60
1000	607
10000	6082
100000	60793
1000000	607925
2000000	1215876
3000000	1823772
4000000	2431735
5000000	3039632
6000000	3647556
7000000	4255503
8000000	4863402
9000000	5471341
10000000	6079290
20000000	12518574
30000000	18237828
$\vdots$	$\vdots$

2.14 Odd Sieve

Definition 2.35. All odd numbers that are not equal to the fractional of two primes.

Observation 2.36. The difference between an odd number and an even number is an odd number; indeed $(2k_1+1)-2k_2 = 2(k_1-k_2)+1$. The number 2 is the only even prime number, all the other primes are odd. Then the difference between a prime number and 2 is always an odd number.

The series generation algorithm, [Le and Smarandache, 1999], with the property from Definition 2.35 is:

1. Take all prime numbers up to the limit L.

2. From every prime number subtract 2. This series becomes a temporary series.

3. Eliminate all numbers that are on the temporary list from the odd numbers list.

The list of the odd numbers that are not the difference of two prime numbers up to the limit $L = 150$ is: 7, 13, 19, 23, 25, 31, 33, 37, 43, 47, 49, 53, 55, 61, 63, 67, 73, 75, 79, 83, 85, 89, 91, 93, 97, 103, 109, 113, 115, 117, 119, 121, 123, 127, 131, 133, 139, 141, 143, 145 .

The length of the odd-numbered series that are not the difference of two prime numbers up to the limit 10, 10^2, 10^3, 10^4, 10^5, 10^6 and 10^7 is, respectively:1, 25, 333, 3772, 40409, 421503, 4335422 .

2.15 n – ary Power Sieve

The list of the odd numbers that are not the difference of two prime numbers up to the limit L, we delete all the n-th term, from the remaining series, we delete all the n^2-th term, and so on until possible.

Program 2.37. Program for generating the series up to the limit L.

$$nPS(L,n) := \left| \begin{array}{l} for\ j \in 1..L \\ \quad S_j \leftarrow j \\ for\ k \in 1..floor(\log(L,n)) \\ \quad \left| \begin{array}{l} break\ \ if\ n^k > last(S) \\ for\ j \in 1..floor\left(\frac{last(S)}{n^k}\right) \\ \quad S_{j\cdot n^k} \leftarrow 0 \end{array} \right. \end{array} \right.$$

$$
\begin{array}{|l}
\quad \begin{array}{|l}
i \leftarrow 1 \\
for\ j \in 1..last(S) \\
\quad if\ S_j \neq 0 \\
\quad\quad \begin{array}{|l} Q_i \leftarrow S_j \\ i \leftarrow i+1 \end{array} \\
S \leftarrow Q \\
Q \leftarrow 0
\end{array} \\
return\ S
\end{array}
$$

The series for $L = 135$ and $n = 2$ is: 1, $\boxed{3}$, $\boxed{5}$, 9, $\boxed{11}$, $\boxed{13}$, $\boxed{17}$, 21, 25, 27, $\boxed{29}$, 33, 35, $\boxed{37}$, $\boxed{43}$, 49, 51, $\boxed{53}$, 57, $\boxed{59}$, 65, $\boxed{67}$, 69, $\boxed{73}$, 75, 77, 81, 85, $\boxed{89}$, 91, $\boxed{97}$, $\boxed{101}$, $\boxed{107}$, $\boxed{109}$, $\boxed{113}$, 115, 117, 121, 123, 129, $\boxed{131}$, 133, where the numbers that appear in the box are primes. To obtain this list we call $nPS(135,2) =$, where nPS is the program 2.37.

The length of the series for $L = 10$, $L = 10^2$, $\ldots$, $L = 10^6$, respectively, is: 4 (2 primes), 31 (14 primes), 293 (97 primes), 2894 (702 primes), 28886 (5505 primes), 288796 (45204 primes) .

The series for $L = 75$ and $n = 3$ is: 1, $\boxed{2}$, 4, $\boxed{5}$, $\boxed{7}$, 8, 10, $\boxed{11}$, 14, 16, $\boxed{17}$, $\boxed{19}$, 20, 22, $\boxed{23}$, 25, 28, $\boxed{29}$, $\boxed{31}$, 32, 34, 35, $\boxed{37}$, 38, $\boxed{41}$, $\boxed{43}$, 46, $\boxed{47}$, 49, 50, 52, 55, 56, 58, $\boxed{59}$, $\boxed{61}$, 62, 64, 65, 68, 70, $\boxed{71}$, $\boxed{73}$, 74 , where the numbers that appear in the box are primes. To obtain this list, we call $nPS(75,3) =$, where nPS is the program 2.37.

The length of the series for $L = 10$, $L = 10^2$, $\ldots$, $L = 10^6$, respectively, is: 7 (3 primes), 58 (20 primes), 563 (137 primes), 5606 (1028 primes), 56020 (8056 primes), 560131 (65906 primes) .

The series for $L = 50$ and $n = 5$ is: 1, $\boxed{2}$, $\boxed{3}$, 4, 6, $\boxed{7}$, 8, 9, $\boxed{11}$, 12, $\boxed{13}$, 14, 16, $\boxed{17}$, 18, $\boxed{19}$, 21, 22, $\boxed{23}$, 24, 26, 27, 28, $\boxed{29}$, 32, 33, 34, 36, $\boxed{37}$, 38, 39, $\boxed{41}$, 42, $\boxed{43}$, 44, 46, $\boxed{47}$, 48, 49 . To obtain this list, we call $nPS(50,5) =$, where nPS is the program 2.37.

The length of the series for $n = 5$ and $L = 10$, $L = 10^2$, $\ldots$, $L = 10^6$, respectively, is: 8 (3 primes), 77 (23 primes), 761 (161 primes), 7605 (1171 primes), 76037 (9130 primes), 760337 (74631 primes) .

For counting the primes, we used the Smarandache primality test, [Cira and Smarandache, 2014].

Conjectures:

1. There are an infinity of primes that belong to this sequence.

2. There are an infinity of numbers of this sequence which are not prime.

2.16 k – ary Consecutive Sieve

The series of positive integers to the imposed limit L, we delete all the k-th term ($k \geq 2$), from the remaining series we delete all the $(k+1)$-th term, and so on until possible„ [Le and Smarandache, 1999].

Program 2.38. The program for generating the series to the limit L.

```
kConsS(L,k) := | for j ∈ 1..L
               |    S_j ← j
               | for n ∈ k..L
               |   | break if n > last(S)
               |   | for j ∈ 1..floor(last(S)/n)
               |   |    S_{j·n} ← 0
               |   | i ← 1
               |   | for j ∈ 1..last(S)
               |   |    if S_j ≠ 0
               |   |      | Q_i ← S_j
               |   |      | i ← i + 1
               |   | S ← Q
               |   | Q ← 0
               | return S
```

The series for $k = 2$ and $L = 10^3$ is: 1, $\boxed{3}$, $\boxed{7}$, $\boxed{13}$, $\boxed{19}$, 27, 39, 49, 63, $\boxed{79}$, 91, $\boxed{109}$, 133, 147, $\boxed{181}$, 207, $\boxed{223}$, 253, 289, $\boxed{307}$, $\boxed{349}$, 387, 399, 459, 481, 529, 567, $\boxed{613}$, 649, $\boxed{709}$, 763, 807, 843, 927, 949, where the numbers that appear in a box are primes. This series was obtained by the call $s := kConsS(10^3, 2)$.

The length of the series for $k = 2$ and $L = 10$, $L = 10^2$, …, $L = 10^6$, respectively, is: 3 (2 primes), 11 (5 primes), 35 (12 primes), 112 (35 primes), 357 (88 primes), 1128 (232 primes) .

The series for $k = 3$ and $L = 500$ is: 1, $\boxed{2}$, 4, $\boxed{7}$, 10, 14, 20, 25, 32, 40, 46, 55, $\boxed{67}$, 74, 91, 104, 112, $\boxed{127}$, 145, 154, 175, 194, 200, 230, $\boxed{241}$, 265, 284, $\boxed{307}$, 325, 355, 382, 404, 422, 464, 475, where the numbers that appear in a box are primes. This series was obtained by the call $s := kConsS(500, 3)$.

The length of the series for $k = 3$ and $L = 10$, $L = 10^2$, …, $L = 10^6$, respectively, is: 5 (2 primes), 15 (3 primes), 50 (10 primes), 159 (13 primes), 504 (30 primes), 1595 (93 primes) . To count the primes, we used Smarandache primality test, [Cira and Smarandache, 2014].

The series for $k = 5$ and $L = 300$ is: 1, $\boxed{2}$, $\boxed{3}$, 4, 6, 8, $\boxed{11}$, $\boxed{13}$, $\boxed{17}$, 21, 24, 28, 34, 38, 46, $\boxed{53}$, 57, 64, $\boxed{73}$, 78, 88, 98, $\boxed{101}$, 116, 121, 133, 143, 154, $\boxed{163}$, 178, 192, 203, 212, $\boxed{233}$, 238, 253, 274, 279, 298, where the numbers that appear in a box are primes. This series was obtained by the call $s := kConsS(300, 5)$.

The length of the series for $k=5$ and $L=10$, $L=10^2$, ..., $L=10^6$, respectively, is: 6 (2 primes), 22 (7 primes), 71 (19 primes), 225 (42 primes), 713 (97 primes), 2256 (254 primes) . To count the primes, we used Smarandache primality test, [Cira and Smarandache, 2014].

2.17 Consecutive Sieve

From the series of positive natural numbers, we eliminate the terms given by the following algorithm. Let $k \geq 1$ and $i = k$. Starting with the element k we delete the following i terms. We do $i = i+1$ and $k = k+i$ and repeat this step as many times as possible.

Program 2.39. Program for generating the series specified by the above algorithm.

```
ConsS(L,k) := | for j ∈ 1..L
              |    S_j ← j
              | i ← k
              | while k ≤ L
              |   | for j ∈ 1..i
              |   |    S_{k+j} ← 0
              |   | i ← i+1
              |   | k ← k+i
              | i ← 1
              | for j ∈ 1..last(S)
              |    if S_j ≠ 0
              |     | Q_i ← S_j
              |     | i ← i+1
              | return Q
```

The call of program 2.39 by command $ConsS(700,1)$ generates the series: 1, $\boxed{3}$, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91, 105, 120, 136, 153, 171, 190, 210, 231, 253, 276, 300, 325, 351, 378, 406, 435, 465, 496, 528, 561, 595, 630, 666, which has only one prime number.

The length of the series for $k=1$ and $L=10$, $L=10^2$, ..., $L=10^6$, respectively, is: 4 (1 prime), 13 (1 prime), 44 (1 prime), 140 (1 prime), 446 (1 prime), 1413 (1 prime) .

The call of program 2.39 by command $ConsS(700,2)$ generates the series: 1, $\boxed{2}$, $\boxed{5}$, 9, 14, 20, 27, 35, 44, 54, 65, 77, 90, 104, 119, 135, 152, 170, 189, 209, 230, 252, 275, 299, 324, 350, 377, 405, 434, 464, 495, 527, 560, 594, 629, 665, which has 2 primes.

The length of the series for $k = 2$ and $L = 10$, $L = 10^2$, ..., $L = 10^6$, respectively, is: 4 (2 primes), 13 (2 primes), 44 (2 primes), 140 (2 primes), 446 (2 primes), 1413 (2 primes) .

For counting the primes, we used the Smarandache primality test, [Cira and Smarandache, 2014].

2.18 Prime Part

2.18.1 Inferior and Superior Prime Part

We consider the function $ipp: [2,\infty) \to \mathbb{N}$, $ipp(x) = p$, where p is the biggest prime number p, $p < x$.

Using the list of primes up to 10^7 generated by program 1.1 in the vector $prime$, we can write a program for the function *ipp*.

Program 2.40. The program for function *ipp*.

$$ipp(x) := \left| \begin{array}{l} return\ \text{"}undefined\text{"}\ \ if\ x < 2 \vee x > 10^7 \\ for\ k \in 1..last(prime) \\ \quad break\ \ if\ x < prime_k \\ return\ \ prime_{k-1} \end{array} \right.$$

For $n = 2, 3, \ldots, 100$ the values of the function *ipp* are: 2, 3, 3, 5, 5, 7, 7, 7, 7, 11, 11, 13, 13, 13, 13, 17, 17, 19, 19, 19, 19, 23, 23, 23, 23, 23, 23, 29, 29, 31, 31, 31, 31, 31, 31, 37, 37, 37, 37, 41, 41, 43, 43, 43, 43, 47, 47, 47, 47, 47, 47, 53, 53, 53, 53, 53, 53, 59, 59, 61, 61, 61, 61, 61, 61, 67, 67, 67, 67, 71, 71, 73, 73, 73, 73, 73, 73, 79, 79, 79, 79, 83, 83, 83, 83, 83, 83, 89, 89, 89, 89, 89, 89, 89, 89, 97, 97, 97, 97 . The graphic of the function on the interval $[2,100)$ is given in the Figure 2.6.

Another program for the function *ipp* is based on the Smarandache primality criterion, [Cira and Smarandache, 2014].

Program 2.41. The program for function *ipp* using he Smarandache primality test (program 1.5).

$$ipp(x) := \left| \begin{array}{l} return\ \text{"}nedefined\text{"}\ \ if\ x < 2 \vee x > 10^7 \\ for\ k \in floor(x)..1 \\ \quad return\ k\ \ if\ TS(k){=}1 \\ return\ \text{"}Error.\text{"} \end{array} \right.$$

We consider the function $spp: [1,\infty) \to \mathbb{N}$, $spp(x) = p$, where p is the smallest prime number p, $p \geq x$.

Using the list of primes up to 10^7 generated by program 1.1 in the vector $prime$, we can write a program for the function *spp*, see Figure 2.8.

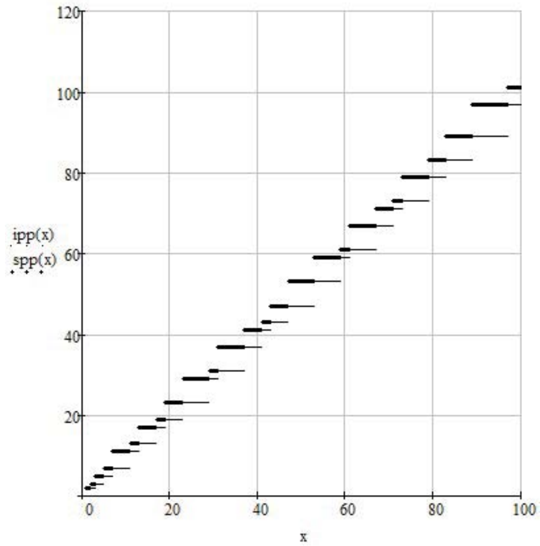

Figure 2.6: Function *ipp* and *spp*

Program 2.42. Program for the function *spp*.

$$spp(x) := \left| \begin{array}{l} return\ \text{"}undefined\text{"}\ \ if\ \ x < 1 \lor x > 10^7 \\ for\ \ k \in 1..last(prime) \\ \quad break\ \ if\ \ x \le prime_k \\ return\ \ prime_k \end{array} \right.$$

For $n = 1, 2, \ldots, 100$ the values of the function *spp* are: 2, 2, 3, 5, 5, 7, 7, 11, 11, 11, 11, 13, 13, 17, 17, 17, 17, 19, 19, 23, 23, 23, 23, 29, 29, 29, 29, 29, 29, 31, 31, 37, 37, 37, 37, 37, 37, 41, 41, 41, 41, 43, 43, 47, 47, 47, 47, 53, 53, 53, 53, 53, 53, 59, 59, 59, 59, 59, 59, 61, 61, 67, 67, 67, 67, 67, 67, 71, 71, 71, 71, 73, 73, 79, 79, 79, 79, 79, 79, 83, 83, 83, 83, 89, 89, 89, 89, 89, 89, 97, 97, 97, 97, 97, 97, 97, 97, 101, 101, 101 . The graphic of the function on the interval $[1, 100)$ in the Figure 2.6.

Program 2.43. Program for the function *spp* using Smarandache primality test (program 1.5).

$$spp(x) := \left| \begin{array}{l} return\ \text{"}nedefined\text{"}\ \ if\ \ x < 1 \lor x > 10^7 \\ for\ \ k \in ceil(x)..last(S) \\ \quad return\ \ k\ \ if\ \ TS(k)=1 \\ return\ \text{"}Error.\text{"} \end{array} \right.$$

Aplication 2.44. Determine prime numbers that have among themselves 120 and the length of the gap that contains the number 120.

$$ipp(120) = 113\ , \quad spp(120) = 127\ , \quad spp(120) - ipp(120) = 14\ .$$

Aplication 2.45. Write a program that provides the series of maximal gap up to 10^6. The distance between two consecutive prime numbers is called gap, $g = g_n = g(p_n) = p_{n+1} - p_n$, where $p_n \in \mathbb{P}_{\geq 2}$, for any $n \in \mathbb{N}^*$. The series of maximal gaps is the series $\{g_n\}$ with property $g_n > g_k$, for any $n > k$, is: 1, 2, 4, 6, 8, 14, 18, ..., where $1 = 3-2$, $2 = 5-3$, $4 = 11-7$, $6 = 29-23$, $8 = 97-89$, $14 = 127-113$, $18 = 541-523$, ...

```
Sgm(L) := | s ← ( 2  1 )
          |       ( 3  2 )
          | k ← 2
          | n ← 4
          | while n ≤ L
          |   | p_i ← ipp(n)
          |   | p_s ← spp(n)
          |   | g ← p_s − p_i
          |   | if g > s_{k,2}
          |   |   | k ← k + 1
          |   |   | s_{k,1} ← p_i
          |   |   | s_{k,2} ← g
          |   | n ← p_s + 1
          | return s
```

By the call $mg := Sgm(999983)$ (where 999983 is the biggest prime smaller than 10^6) we get the result:

$$mg^{\mathrm{T}} = \begin{bmatrix} 2 & 3 & 7 & 23 & 89 & 113 & 523 & 887 & 1129 & 1327 & 9551 & 15683 & 19609 & 31397 & 155921 & 360653 & 370261 & 492113 \\ 1 & 2 & 4 & 6 & 8 & 14 & 18 & 20 & 22 & 34 & 36 & 44 & 52 & 72 & 86 & 96 & 112 & 114 \end{bmatrix} \tag{2.91}$$

So far, we know 75 terms of the series of maximal gaps, [Weisstein, 2014f], [Oliveira e Silva, 2014]. The last maximal gap is 1476 (known as of December, 14, 2014), and the inferior prime is 1425172824437699411. To note that getting a new maximal gap is considered an important mathematical result. For determining these gaps, leading researchers were involved, as: Tomás Oliveira e Silva, Donald E. Knuth, Siegfried Herzog, [Oliveira e Silva, 2014].

The series (2.91) can also be obtained directly using series of primes.

2.18.2 Inferior and Superior Fractional Prime Part

Function 2.46. The function *inferior fractional prime part, ppi* : $[2,\infty) \to \mathbb{R}_+$, is defined by the formula (see Figure 2.7):

$$ppi(x) := x - ipp(x)\ .$$

Function 2.47. The function *superior fractional prime part*, *pps* : $[2,\infty) \to \mathbb{R}_+$, is defined by the formula (see Figure 2.7):

$$pps(x) := ssp(x) - x\ .$$

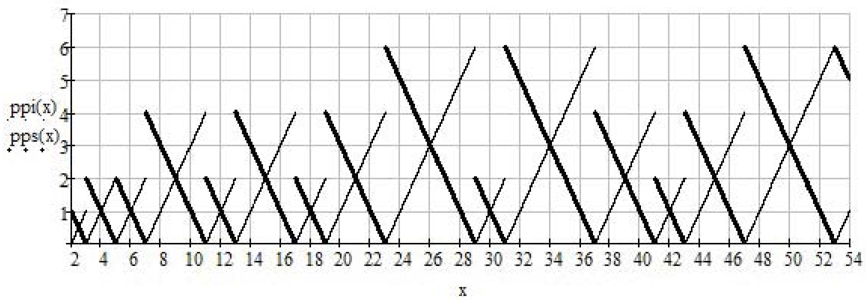

Figure 2.7: The graphic of the functions *ppi* and *pps*

Examples of calls of the functions *ppi* and *pps*:

$$ppi(\pi^3 + e^5) = 0.41943578287637706\ , \quad pps(\pi^3 + e^5) = 1.580564217123623\ .$$

2.19 Square Part

2.19.1 Inferior and Superior Square Part

The functions *isp*, *ssp* : $\mathbb{R}_+ \to \mathbb{N}$, are the inferior square part and respectively the superior square part of the number x, [Popescu and Niculescu, 1996].

Function 2.48. The function *isp* is given by the formula (see Figure 2.8):

$$isp(x) := \left(floor(\sqrt{x})\right)^2\ .$$

Function 2.49. The function *ssp* is given by the formula (see Figure 2.8):

$$ssp(x) := \left(ceil(\sqrt{x})\right)^2\ .$$

Aplication 2.50. To determine between what perfect squares we find the irrational numbers: π^{ϕ}, π^{ϕ^2}, $e^{\phi+1}$, $e^{2\phi+3}$, ϕ^{e^2}, ϕ^{π^3}, $e^{\pi+\phi}$, where ϕ is the golden number $\phi = (1+\sqrt{5})/2$. We find the answer in the Table 2.38.

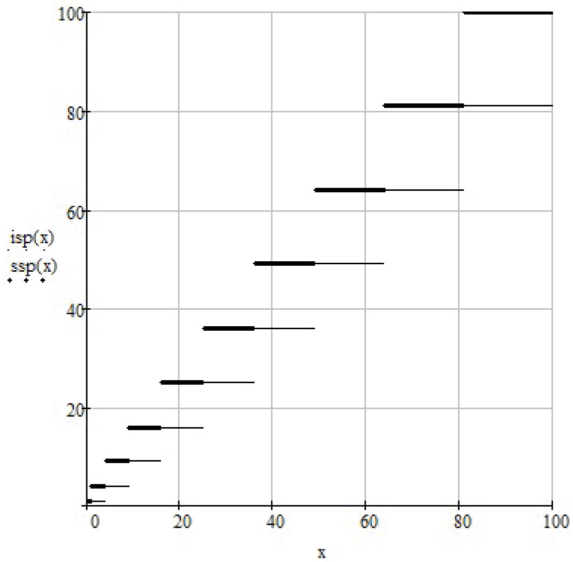

Figure 2.8: The graphic of the functions *isp* and *ssp*

2.19.2 Inferior and Superior Fractional Square Part

Function 2.51. The function *inferior fractional square part, spi* : $\mathbb{R} \to \mathbb{R}_+$, is given by the formula (see Figure 2.9):

$$spi(x) := x - isp(x)\ .$$

Function 2.52. The function *superior fractional square part, sps* : $\mathbb{R} \to \mathbb{R}_+$, is given by the formula (see Figure 2.9):

$$sps(x) := ssp(x) - x\ .$$

Examples of calls of the functions *spi* and *sps*:

$$spi(\pi^3) = 6.006276680299816,\ \ sps(\pi^3 + e^3) = 12.90818639651252\ .$$

2.20 Cubic Part

2.20.1 Inferior and Superior Cubic Part

The functions $icp, scp : \mathbb{R} \to \mathbb{Z}$, are the *inferior cubic part* and respectively the *superior cubic part* of the number x, [Popescu and Seleacu, 1996].

$isp(x)$	x	$ssp(x)$
4	π^{ϕ}	9
16	π^{ϕ^2}	25
9	$e^{\phi+1}$	16
474	$e^{2\phi+3}$	529
25	ϕ^{e^2}	36
3017169	ϕ^{π^3}	3020644
100	$e^{\pi+\phi}$	121

Table 2.38: Applications to functions *isp* and *ssp*

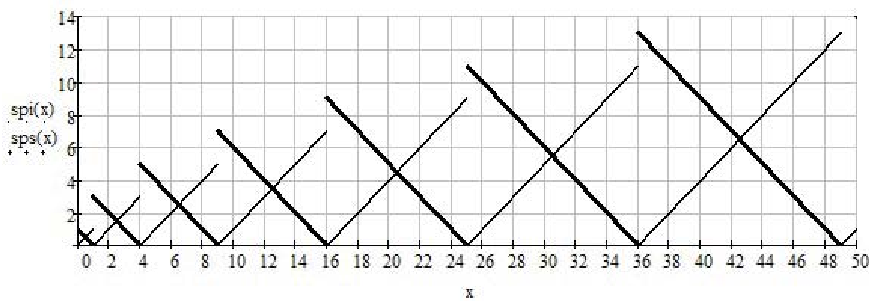

Figure 2.9: The graphic of the functions *spi* and *sps*

Function 2.53. The function *icp* is given by the formula (see Figure 2.10):

$$icp(x) := \left(floor(\sqrt[3]{x})\right)^3 .$$

Function 2.54. The function *scp* is given by the formula (see Figure 2.10):

$$scp(x) := \left(ceil(\sqrt[3]{x})\right)^3 .$$

Aplication 2.55. Determine where between the perfect cubes we find the irrational numbers: $\phi^2+e^3+\pi^4$, $\phi^3+e^4+\pi^5$, $\phi^4+e^5+\pi^6$, $e^{\pi\sqrt{58}}$, $e^{\pi\sqrt{163}}$ where $\phi=(1+\sqrt{5})/2$ is the golden number. The answer is in the Table 2.39. The constants $e^{\sqrt{58}\pi}$ and $e^{\sqrt{163}\pi}$ are related to the results of the noted indian mathematician Srinivasa Ramanujan and we have $262537412640768000=640320^3$, $262538642671796161=640321^3$, $24566036643=2907^3$ and $24591397312=2908^3$. As known the number $e^{\pi\sqrt{163}}$ is an *almost integer* of 640320^3+744 or of $\left(icp(e^{\pi\sqrt{163}})\right)^3+744$ because

$$\left|e^{\pi\sqrt{163}}-(640320^3+744)\right| \approx 7.4992746048967683092367764 2\times 10^{-13} .$$

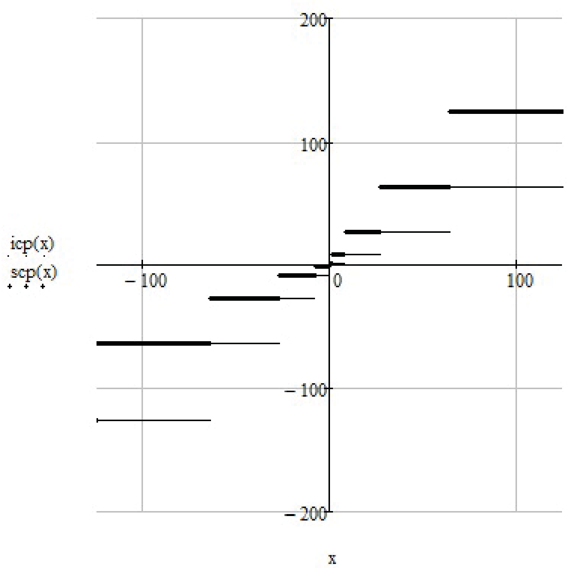

Figure 2.10: The functions *icp* and *scp*

$icp(x)$	x	$scp(x)$
64	$\phi^2+e^3+\pi^4$	125
343	$\phi^3+e^4+\pi^5$	512
1000	$\phi^4+e^5+\pi^6$	1331
24566036643	$e^{\pi\sqrt{58}}$	24591397312
262537412640768000	$e^{\pi\sqrt{163}}$	262538642671796161

Table 2.39: Applications to the functions *icp* and *scp*

2.20.2 Inferior and Superior Fractional Cubic Part

Function 2.56. The *inferior fractional cubic part* function, $cpi:\mathbb{R}\to\mathbb{R}_+$, is defined by the formula (see Figure 2.11):

$$cpi(x) := x - icp(x)\ .$$

Function 2.57. The *superior fractional cubic part* function, $cps:\mathbb{R}\to\mathbb{R}_+$, is given by the formula (see Figure 2.11):

$$cps(x) := scp(x) - x\ .$$

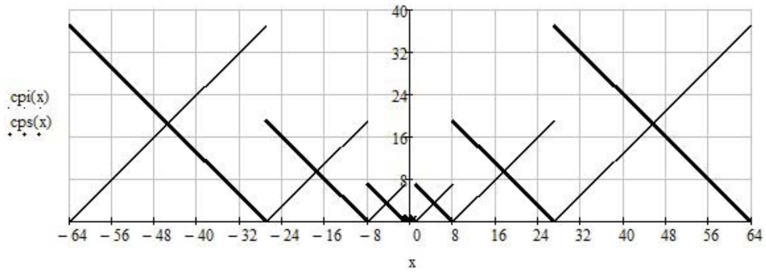

Figure 2.11: The graphic of the functions *cpi* and *cps*

Examples of calling the functions *cpi* and *cps*:

$$cpi\left(\sqrt{\pi^3+\pi^5}\right) = 10.358266842640163\ ,$$

$$cpi\left(e^{\pi\sqrt{163}}\right) = 743.99999999999925007253951\ ,$$

$$cps\left(\sqrt{\pi^3+\pi^5}\right) = 8.641733157359837\ .$$

2.21 Factorial Part

2.21.1 Inferior and Superior Factorial Part

The functions *ifp*, *sfp* : $\mathbb{R}_+ \to \mathbb{N}$, are the *inferior factorial part* and respectively the *superior factorial part* of the number x, [Dumitrescu and Seleacu, 1994].

Program 2.58. Program for function *ifp*.

$$ifp(x) := \left| \begin{array}{l} return\ "undefined"\ \ if\ x<0 \vee x>18! \\ for\ \ k\in 1..18 \\ \quad return\ (k-1)!\ \ if\ x<k! \\ return\ "Err." \end{array} \right.$$

Program 2.59. Program for function sfp.

$$sfp(x) := \left| \begin{array}{l} return\ "undefined"\ \ if\ x<0 \vee x>18! \\ for\ \ k\in 1..18 \\ \quad return\ k!\ \ if\ x<k! \\ return\ "Error." \end{array} \right.$$

Aplication 2.60. Determine in what factorial the numbers $e^{k\pi}$ for $k = 1, 2, \ldots, 11$.

Table 2.40: Factorial parts for $e^{k\pi}$

$6 = 3! > e^{\pi} > 4! = 24$
$120 = 5! > e^{2\pi} > 6! = 720$
$5040 = 7! > e^{3\pi} > 8! = 40320$
$40320 = 8! > e^{4\pi} > 9! = 362880$
$3628800 = 10! > e^{5\pi} > 11! = 39916800$
$39916800 = 11! > e^{6\pi} > 12! = 479001600$
$479001600 = 12! > e^{7\pi} > 13! = 6227020800$
$6227020800 = 13! > e^{8\pi} > 14! = 87178291200$
$1307674368000 = 15! > e^{9\pi} > 16! = 20922789888000$
$20922789888000 = 16! > e^{10\pi} > 17! = 355687428096000$
$355687428096000 = 17! > e^{11\pi} > 18! = 6402373705728000$

Function 2.61. The *inferior factorial difference part, $fpi : \mathbb{R}_+ \to \mathbb{R}_+$*, is defined by the formula:

$$fpi(x) := x - ifp(x)!\ .$$

Function 2.62. The *superior factorial difference part, $fps : \mathbb{R}_+ \to \mathbb{R}_+$*, is given by the formula:

$$fps(x) := sfp(x)! - x\ .$$

Aplication 2.63. Determine the *inferior* and *superior factorial difference parts* for the numbers $e^{k\pi}$ for $k = 1, 2, \ldots, 10$.

Table 2.41: Factorial difference parts for $e^{k\pi}$

$fpi(e^{\pi}) = 17.140692632779263$
$fpi\left(e^{2\pi}\right) = 415.4916555247644$
$fpi\left(e^{3\pi}\right) = 7351.647807916686$
$fpi\left(e^{4\pi}\right) = 246431.31313665299$
$fpi\left(e^{5\pi}\right) = 3006823.9993411247$
$fpi\left(e^{6\pi}\right) = 113636135.39544642$
$fpi\left(e^{7\pi}\right) = 3074319680.8470373$
$fpi\left(e^{8\pi}\right) = 75999294785.594800$
$fpi\left(e^{9\pi}\right) = 595099527292.15620$
$fpi\left(e^{10\pi}\right) = 23108715972631.90$
$fps(e^{\pi}) = 0.85930736722073680$
$fps\left(e^{2\pi}\right) = 184.50834447523562$

Continued on next page

$$\begin{array}{|c|}
fps\left(e^{3\pi}\right) = 27928.352192083315 \\
fps\left(e^{4\pi}\right) = 76128.686863347010 \\
fps\left(e^{5\pi}\right) = 33281176.000658877 \\
fps\left(e^{6\pi}\right) = 325448664.60455360 \\
fps\left(e^{7\pi}\right) = 2673699519.1529627 \\
fps\left(e^{8\pi}\right) = 4951975614.4051970 \\
fps\left(e^{9\pi}\right) = 19020015992707.844 \\
fps\left(e^{10\pi}\right) = 311655922235368.10 \\
\hline
\end{array}$$

2.22 Function Part

Let f be a strictly ascending real on the interval $[a,b]$, where $a,b \in \mathbb{R}$, $a < b$. We can generalize the notion of part (inferior or superior) in relation to the function f, [Castillo, 2014].

2.22.1 Inferior and Superior Function Part

We define the function, $ip : [a,b] \to \mathbb{R}$, *inferior part relative to the function* f.

Program 2.64. Program for the function *ip*. We have to define the function f in relation to which we consider the inferior part function. It remains the responsibility of the user as the function f to be strictly ascending on $[a,b] \subset \mathbb{R}$.

$$ip(f,a,b,x) := \left| \begin{array}{l} return\ "undefined"\ \ if\ \ x < a \vee x > b \\ return\ "a\ or\ b\ not\ integer"\ \ if\ \ a \neq trunc(a) \vee b \neq trunc(b) \\ for\ z \in a..b \\ \quad return\ z-1\ \ if\ \ x < f(z) \\ return\ \ "Error." \end{array} \right.$$

We want to determine the inferior part of e^{π} in relation to the function $f(z) := 2z + \ln(z^2 + z + 1)$ on the interval $[0, 10^6]$. The function is strictly ascending on the interval $[0, 10^6]$ so it makes sense to consider the *inferior part function* in relation to f and we have $ip(f, 0, 10^6, e^{\pi}) = 22.51085950651685$. Other examples

of calling the function *ip*:

$$
\begin{aligned}
g(z) &:= z+\sqrt{z} & ip\left(g,0,10^2,e^{\pi}\right) &= 22.242640687119284\ ,\\
h(z) &:= z+3\arctan(z) & ip\left(h,-6,6,e^{\sqrt{\pi}}\right) &= 5.321446153382271\ ,\\
&& ip\left(h,-6,6,e^{2\sqrt{\pi}}\right) &= "undefined"\ ,\\
&& ip\left(h,-36,36,e^{2\sqrt{\pi}}\right) &= 34.61242599274995\ ,
\end{aligned}
$$

We define the function, $sp:[a,b]\to\mathbb{R}$, *superior part relative to the function* f.

Program 2.65. Program for function *sp*. We have to define the function f related to which we consider the function a superior part. It remains the responsibility of the user as the function f to be strictly increasing $[a,b]\subset\mathbb{R}$.

```
sp(f,a,b,x) := | return "undefined" if x < a ∨ x > b
               | return "a or b not integer" if a ≠ trunc(a) ∨ b ≠ trunc(b)
               | for z ∈ a..b
               |   return z if x < f(z)
               | return "Err."
```

We want to determine the superior part of e^{π} related to the function $f(z):=2z+\ln(z^2+z+1)$ on the interval $[0,10^6]$. The function is strictly ascending on the interval $[0,10^6]$ so it makes sense to consider the part function in relation to f and we have $sp(f,0,10^6,e^{\pi})=24.709530201312333$. Other examples of function *sp*:

$$
\begin{aligned}
g(z) &:= z+\sqrt{z} & sp\left(g,0,10^2,e^{\pi}\right) &= 23.358898943540673\ ,\\
h(z) &:= z+3\arctan(z) & sp\left(h,-6,6,e^{\sqrt{\pi}}\right) &= 6.747137317194763\ ,\\
&& sp\left(h,-6,6,e^{2\sqrt{\pi}}\right) &= "undefined"\ ,\\
&& sp\left(h,-36,36,e^{2\sqrt{\pi}}\right) &= 35.61564833307893\ ,
\end{aligned}
$$

Observation 2.66. All values displayed by functions *ip* and *sp* have an accuracy of mathematical computing, given by software implementation, of 10^{-15}. To obtain better accuracy it is necessary to turn to symbolic computation.

2.22.2 Inferior and Superior Fractional Function Part

The *fractional inferior part* function in relation to the function f, $ipd:[a,b]\subset\mathbb{R}$, is given by the formula $ipd(f,a,b,x):=x-ip(f,a,b,x)$. Before the

call of function *ipd* we have to define the strictly ascending function f on the real interval $[a,b]$. Examples of calls of function *ipd*:

$$f(z) := 2z + \ln(z^2 + z + 1) \qquad ipd\left(f, 0, 10^6, e^{\pi}\right) = 0.6298331262624117\ ,$$
$$g(z) := z + \sqrt{z} \qquad ipd\left(g, 0, 10^2, e^{\pi}\right) = 0.8980519456599794\ ,$$
$$h(z) := z + 3\arctan(z) \qquad ipd\left(h, -6, 6, e^{\sqrt{\pi}}\right) = 0.5638310966357558\ ,$$

The *fractional superior part* function in relation to the function f, spd : $[a,b] \subset \mathbb{R}$, is given by the formula $sdp(f,a,b,x) := sp(f,a,b,x) - x$. As with the function *ipd* before the call of function *spd* we have to define the strictly ascending function f on the real interval $[a,b]$. Examples of calls of function *spd*:

$$f(z) := 2z + \ln(z^2 + z + 1) \qquad spd\left(f, 0, 10^6, e^{\pi}\right) = 1.5688375685330698\ ,$$
$$g(z) := z + \sqrt{z} \qquad spd\left(g, 0, 10^2, e^{\pi}\right) = 0.21820631076140984\ ,$$
$$h(z) := z + 3\arctan(z) \qquad spd\left(h, -6, 6, e^{\sqrt{\pi}}\right) = 0.8618600671767362\ ,$$

The remark taken in Observation 2.66 is valid for the functions *ipd* and *spd*.

2.23 Smarandache type Functions

2.23.1 Smarandache Function

The function that associates to each natural number n the smallest natural number m which has the property that $m!$ is a multiple of n was considered for the first time by Lucas [1883]. Other authors who have considered this function in their works are: Neuberg [1887], Kempner [1918]. This function was rediscovered by Smarandache [1980].

Therefore, function $S : \mathbb{N}^* \to \mathbb{N}^*$, $S(n) = m$, where m is the smallest natural that has the property that n divides $m!$, (or $m!$ is a multiple of n) is known in the literature as *Smarandache's function*, [Hazewinkel, 2011], [DeWikipedia, 2015, 2013]. The values of the function, for $n = 1, 2, \ldots, 18$, are: 1, 2, 3, 4, 5, 3, 7, 4, 6, 5, 11, 4, 13, 7, 5, 6, 17, 6 obtained by means of an algorithm that results from the definition of function S, as follows:

Program 2.67.

$$S(n) = \left| \begin{array}{l} for\ \ m = 1..n \\ \quad return\ m\ \ if \quad \bmod(m!, n) = 0 \end{array} \right.$$

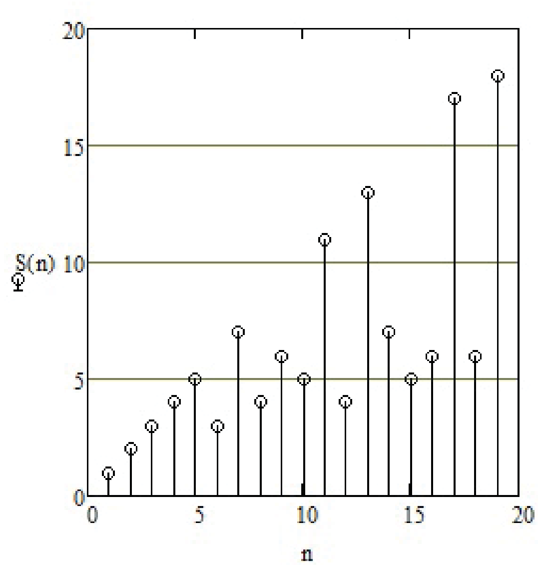

Figure 2.12: S function

Properties of the function S

1. $S(1) = 1$, where it should be noted that [Sloane, 2014, A002034] defines $S(1) = 1$, while Ashbacher [1995] and [Russo, 2000, p. 4] take $S(1) = 0$;

2. $S(n!) = n$, [Sondow and Weisstein, 2014];

3. $S(p^m) = m \cdot p$, where $p \in \mathbb{P}_{\geq 2}$ and $1 \leq m < p$, [Kempner, 1918]; Particular case: $m = 1$, then $S(p) = p$, for all $p \in \mathbb{P}_{\geq 2}$;

4. $S(p_1 \cdot p_2 \cdots p_m) = p_m$, where $p_1 < p_2 < \ldots < p_m$ and $p_k \in \mathbb{P}_{\geq 2}$, for all $k \in \mathbb{N}^*$, [Sondow and Weisstein, 2014];

5. $S(p^{p^m}) = p^{m+1} - p^m + 1$, for all $p \in \mathbb{P}_{\geq 2}$, [Ruiz, 1999b];

6. $S(P_p) = M_p$, if P_p is the pth even *perfect number* and M_p is the corresponding *Mersenne prime*, [Ashbacher, 1997, Ruiz, 1999a].

2.23.2 Smarandache Function of Order k

Definition 2.68. The function $S_k : \mathbb{N}^* \to \mathbb{N}^*$, $m = S_k(n)$ is the smallest integer m such that

$$n \mid m \underbrace{!! \ldots !}_{k\ times} \quad \text{or} \quad n \mid kf(m,k)\ .$$

The function multifactorial kf is given by 2.15. For $k = 1$ we can say that $S_1(n) = S(n)$, given by 2.67.

Program 2.69. for calculating the values of the Smarandache function of k rank.

$$S(n,k) := \left| \begin{array}{l} for\ m \in 1..n \\ \quad return\ m\ if\ \ \bmod(kf(m,k),n) = 0 \end{array} \right.$$

which uses the function kf given by 2.15.

To display the first 30 values of the functions S we use the sequence of commands: $n = 1..30$ $S(n,2) \rightarrow$ and $S(n,3) \rightarrow$.

1. The values of the function $S_2(n)$ are: 1, 2, 3, 4, 5, 6, 7, 4, 9, 10, 11, 6, 13, 14, 5, 6, 17, 12, 19, 10, 7, 22, 23, 6, 15, 26, 9, 14, 29, 10 .

2. The values of the function $S_3(n)$ are: 1, 2, 3, 4, 5, 6, 7, 8, 6, 5, 11, 12, 13, 7, 15, 8, 17, 6, 19, 8, 21, 11, 23, 12, 20, 13, 9, 7, 29, 15 .

Properties of the function S_k, $k > 1$

The question is whether we have the same set of properties as the Smarandache function S_k?

1. $S_k(1) = 1$, may be taken by convention;

2. $S_2(n!!) = n$ and $S_3(n!!!) = n$, result from definition;

3. $S_k(p^\alpha) = [k \cdot \alpha - (k-1)]p$, where $p \in \mathbb{P}_{\geq 3}$ and $2 \leq \alpha < p$;

   (a) If $k = 2$

   $$p^\alpha \mid 1 \cdots p \cdots 3p \cdots 5p \cdots (2\alpha - 1)p = [(2\alpha - 1)p]!!$$

   and therefore
   $$S_2(p^\alpha) = (2\alpha - 1)p\ .$$

   (b) If $k = 3$

   $$p^\alpha \mid 1 \cdots p \cdots 7p \cdots 13p \cdots (3\alpha - 2)p = [(3\alpha - 2)p]!!!$$

   and therefore
   $$S_3(p^\alpha) = (3\alpha - 2)p\ .$$

   In the case: $\alpha = 1$, then $S_k(p^\alpha) = S_k(p) == p$, for $k > 1$ and all $p \in \mathbb{P}_{\geq 2}$;

4. $S_2(2\cdot p_1\cdot p_2\cdots p_m)=2\cdot p_m$? Where $p_1<p_2<\ldots<p_m$ and $p_k\in\mathbb{P}_{\geq 3}$, for all $k\in\mathbb{N}^*$?

5. $S_2(3\cdot p_1\cdot p_2\cdots p_m)=3\cdot p_m$? Where $p_1<p_2<\ldots<p_m$ and $p_k\in\mathbb{P}_{\geq 5}$, for all $k\in\mathbb{N}^*$?

6. $S_3(p_1\cdot p_2\cdots p_m)=2\cdot p_m$? Where $p_1<p_2<\ldots<p_m$, $p_1\neq 3$, and $p_k\in\mathbb{P}_{\geq 2}$, for all $k\in\mathbb{N}^*$?

7. $S_3(3\cdot p_1\cdot p_2\cdots p_m)=3\cdot p_m$? Where $p_1<p_2<\ldots<p_m$ and $p_k\in\mathbb{P}_{\geq 5}$, for all $k\in\mathbb{N}^*$?

2.23.3 Smarandache–Cira Function of Order k

Function $SC\colon\mathbb{N}^*\times\mathbb{N}^*\to\mathbb{N}^*$, $m=SC(n,k)$ is the smallest integer m such that $n\mid 1^k\cdot 2^k\cdots m^k$ (or $n\mid(m!)^k$). For $k=1$ we can say that $S(n,1)=S(n)$, given by 2.67.

Program 2.70. for calculating the values of Smarandache–Cira function of k rank.

$$SC(n,k):=\left|\begin{array}{l} for\ m\in 1..n \\ \quad return\ m\ \ if\quad \bmod\left((m!)^k,n\right)=0\end{array}\right.$$

The values given by the function $SC(n,1)$ The values given by the function Smarandache S, 2.67.

To display the first 113 values of the functions SC we use the sequence of commands: $n=1..113$, $SC(n,2)\to$ and $SC(n,3)\to$.

1. The values of the function $SC(n,2)$ are: 1, 2, 3, 2, 5, 3, 7, 4, 3, 5, 11, 3, 13, 7, 5, 4, 17, 3, 19, 5, 7, 11, 23, 4, 5, 13, 6, 7, 29, 5, 31, 4, 11, 17, 7, 3, 37, 19, 13, 5, 41, 7, 43, 11, 5, 23, 47, 4, 7, 5, 17, 13, 53, 6, 11, 7, 19, 29, 59, 5, 61, 31, 7, 4, 13, 11, 67, 17, 23, 7, 71, 4, 73, 37, 5, 19, 11, 13, 79, 5, 6, 41, 83, 7, 17, 43, 29, 11, 89, 5, 13, 23, 31, 47, 19, 4, 97, 7, 11, 5, 101, 17, 103, 13, 7, 53, 107, 6, 109, 11, 37, 7, 113 .

2. The values of the function $SC(n,3)$ are: 1, 2, 3, 2, 5, 3, 7, 2, 3, 5, 11, 3, 13, 7, 5, 4, 17, 3, 19, 5, 7, 11, 23, 3, 5, 13, 3, 7, 29, 5, 31, 4, 11, 17, 7, 3, 37, 19, 13, 5, 41, 7, 43, 11, 5, 23, 47, 4, 7, 5, 17, 13, 53, 3, 11, 7, 19, 29, 59, 5, 61, 31, 7, 4, 13, 11, 67, 17, 23, 7, 71, 3, 73, 37, 5, 19, 11, 13, 79, 5, 6, 41, 83, 7, 17, 43, 29, 11, 89, 5, 13, 23, 31, 47, 19, 4, 97, 7, 11, 5, 101, 17, 103, 13, 7, 53, 107, 3, 109, 11, 37, 7, 113 .

2.24 Smarandache–Kurepa Functions

2.24.1 Smarandache–Kurepa Function of Order 1

We notation

$$\Sigma_1(n) = \sum_{k=1}^{n} k!\ . \tag{2.92}$$

Program 2.71. for calculating the sum (2.92), (2.94) and 2.96.

$$\Sigma(k,n) := \left| \begin{array}{l} s \leftarrow 0 \\ for\ \ j \in 1..n \\ \quad s \leftarrow s + kf(j,k) \\ return\ s \end{array} \right.$$

The program uses the subprogram *kf*, 2.15.

With commands $n := 1..20$ and $\Sigma(1,n) \rightarrow$, result first 20 values of the function Σ_1:

$$\begin{aligned} &1, 3, 9, 33, 153, 873, 5913, 46233, 409113, 4037913, 43954713, \\ &\quad 522956313, 6749977113, 93928268313, 1401602636313, 22324392524313, \\ &\qquad 378011820620313, 6780385526348313, 128425485935180313, \\ &\qquad\qquad 2561327494111820313\ . \end{aligned}$$

Definition 2.72 ([Mudge, 1996a,b, Ashbacher, 1997])**.** The function $SK_1 : \mathbb{P}_{\geq 2} \to \mathbb{N}^*$, $m = SK_1(p)$ is the smallest $m \in \mathbb{N}^*$ such that $p \mid [1 + \Sigma_1(m-1)]$.

Proposition 2.73. *If* $p \nmid [1 + \Sigma_1(m-1)]$, *for all* $m \leq p$, *then* p *never divides any sum for all* $m > p$ *.*

Proof. If $p \nmid [1 + \Sigma_1(m-1)]$, for all $m \leq p$, then $1 + \Sigma_1(p-1) = \mathcal{M} \cdot p + r$, with $1 \leq r < p$.

Let $m > p$, then

$$\begin{aligned} 1 + \Sigma_1(m-1) &= 1 + \Sigma_1(p-1) + p! + (p+1)! + \ldots + (m-1)! \\ &= \mathcal{M} \cdot p + r + p! + (p+1)! + \ldots + (m-1)! \\ &= [\mathcal{M} + (p-1)!\big(1 + (p+1) + \ldots + (p+1)\cdots(m-1)\big)]\,p + r \\ &= \mathcal{M} \cdot p + r\ , \end{aligned}$$

then $p \nmid [1 + \Sigma_1(m-1)]$ for all $m > p$. □

Program 2.74. for calculating the values of functions SK_1, SK_2 and SK_3.

$$SK(k,p):=\left|\begin{array}{l} for\ m\in 2..k\cdot p-1\\ \quad return\ \ m\ \ if\ \ \ \mathrm{mod}\,(1+\Sigma(k,m-1),p)=0\\ return\ \ -1\end{array}\right.$$

The program uses the subprogram Σ, 2.71, and the utilitarian function Mathcad *mod*.

With commands $k := 1..25$ and $SK(1, prime_k) \rightarrow$ are obtained first 25 values of the function SK_1:

p	2	3	5	7	11	13	17	19	23	29	31	37
$SK_1(p)$	2	−1	4	6	6	−1	5	7	7	−1	12	22

41	43	47	53	59	61	67	71	73	79	83	89	97
16	−1	−1	−1	−1	55	−1	54	42	−1	−1	24	−1

(2.93)

If $SK_1(p) = -1$, then for p the function SK_1 is undefined, [Weisstein, 2015h].

2.24.2 Smarandache–Kurepa Function of order 2

We notation

$$\Sigma_2(n) = \sum_{k=1}^{n} k!! \,. \tag{2.94}$$

With commands $n := 1..20$ and $\Sigma(2,n) \rightarrow$, (using the program 2.71) result first 20 values of the function Σ_2:

$$1,3,6,14,29,77,182,566,1511,5351,15746,61826,\\ 196961,842081,2869106,13191026,47650451,233445011,\\ 888174086,4604065286\,.$$

Definition 2.75. The function $SK_2 : \mathbb{P}_{\geq 2} \rightarrow \mathbb{N}^*$, $m = SK_2(p)$ is the smallest $m \in \mathbb{N}^*$ such that $p \mid [1+\Sigma_2(m-1)]$.

Proposition 2.76. *If $p \nmid [1+\Sigma_2(m-1)]$, for all $m \leq 2p$, then p never divides any sum for all $m > 2p$*

Proof. If for all m, $m \leq 2p$, $p \nmid [1+\Sigma_2(m-1)]$, then $p \nmid [1+\Sigma_2(2p-1)]$ i.e. $1+\Sigma_2(2p-1) = \mathscr{M}\cdot p + r$, with $1 \leq r < p$.

Let $m=2p+1$,

$$
\begin{aligned}
1+\Sigma_2(m-1) &= 1+\Sigma_2(2p-1)+2p!! \\
&= \mathscr{M}\cdot p+r+2\cdot 4\cdots(p-1(p+1)\cdots 2p \\
&= [\mathscr{M}+2\cdot 4\cdots(p-1)(p+1)\cdots 2]\,p+r=\mathscr{M}\cdot p+r\ ,
\end{aligned}
$$

then $1+\Sigma_2(2p)=\mathscr{M}p+r$, with $1\le r<p$.

Let $m=2p+2$ and using the above statement, we have that

$$
\begin{aligned}
1+\Sigma_2(m-1) &= 1+\Sigma_2(2p)+(2p+1)!! \\
&= \mathscr{M}\cdot p+r+1\cdot 3\cdots(p-2)p(p+2)\cdots(2p+1) \\
&= [\mathscr{M}+1\cdot 3\cdots(p-2)(p+2)\cdots(2p+1)]\,p+r=\mathscr{M}\cdot p+r\ ,
\end{aligned}
$$

then $1+\Sigma_2(2p+1)=\mathscr{M}p+r$, with $1\le r<p$.

Through complete induction, it follows that $p\nmid\Sigma_2(m)$, for all $m>2p$. □

With commands $k:=1..25$ and $SK(2,prime_k)\to$ (using the program 2.74) are obtained first 25 values of the function SK_2:

p	2	3	5	7	11	13	17	19	23	29	31	37
$SK_2(p)$	2	5	5	4	−1	7	14	−1	31	12	17	13

41	43	47	53	59	61	67	71	73	79	83	89	97
−1	−1	20	43	40	8	17	50	17	−1	−1	46	121

. (2.95)

If $SK_2(p)=-1$, then for p function SK_2 is undefined.

2.24.3 Smarandache–Kurepa Function of Order 3

We notation

$$\Sigma_3(n)=\sum_{k=1}^{n}k!!!\ . \tag{2.96}$$

With commands $n:=1..20$ and $\Sigma(3,n)\to$, (using the program 2.71) result first 20 values of the function Σ_3:

$$1,3,6,10,20,38,66,146,308,588,1468,3412,7052,$$
$$19372,48532,106772,316212,841092,1947652,6136452\ .$$

Definition 2.77. The function $SK_3:\mathbb{P}_{\ge 2}\to\mathbb{N}^*$, $m=SK_3(p)$ is the smallest $m\in\mathbb{N}^*$ such that $p\mid[1+\Sigma_3(m-1)]$.

Theorem 2.78. *Let $p\in\mathbb{P}_{\ge 5}$, then exists $k_1,k_2\in\{0,1,\ldots,p-1\}$ for which* $\mod(3k_1+1,p)=0$ *and* $\mod(3k_2+1,p)=0$.

Proof. If $p \in \mathbb{P}_{\geq 5}$, then $M = \{0, 1, 2, ..., p-1\}$ is a complete system of residual classes $(mod\ p)$. Let q be a relative prime with p, (i.e. $gcd(p, q) = 1$), then $q \cdot M$ is also a complete system of residual classes $(mod\ p)$ and $M = q \cdot M$, [Smarandache, 1999a, T 1.14].

Because 3 is relative prime with p, then there exists $k_1 \in 3 \cdot M$ such that $\bmod(3k_1, p-1) = 0$, i.e. $\bmod(3k_1+1, p) = 0$. Also, there exists $k_2 \in 3 \cdot M$ such that $\bmod(3k_2, p-2) = 0$, i.e. $\bmod(3k_2+2, p) = 0$. □

Proposition 2.79. *If $p \nmid [1 + \Sigma_3(m-1)]$, for all $m \leq 3p$, then p never divides any sum for all $m > 3p$.*

Proof. If for all m, $m \leq 3p$, $p \nmid [1 + \Sigma_3(m-1)]$, then $p \nmid [1 + \Sigma_3(3p-1)]$ i.e. $1 + \Sigma_3(3p-1) = \mathcal{M} \cdot p + r$, with $1 \leq r < p$.

Let $m = 3p+1$,

$$
\begin{aligned}
1 + \Sigma_3(m-1) = 1 + \Sigma_3(3p-1) + 3p!!! &= \mathcal{M} \cdot p + r + 3 \cdot 6 \cdots 3(p-1) \cdot 3p \\
&= [\mathcal{M} + 3 \cdot 6 \cdots 3(p-1) \cdot 3] p + r = \mathcal{M} \cdot p + r\ ,
\end{aligned}
$$

then $1 + \Sigma_3(3p) = \mathcal{M} p + r$, with $1 \leq r < p$.

Let $m = 3p+2$ and using the above statement, we have that

$$
\begin{aligned}
1 + \Sigma_3(m-1) &= 1 + \Sigma_3(3p) + (3p+1)!!! \\
&= \mathcal{M} \cdot p + r + 1 \cdot 4 \cdots \alpha p \cdots (3p+1) \\
&= [\mathcal{M} + 1 \cdot 4 \cdots \alpha \cdots (3p+1)] p + r = \mathcal{M} \cdot p + r\ ,
\end{aligned}
$$

because exists k, according to Theorem 2.78, $k \in \{0, 1, \ldots, p-1\}$, for which $3k + 1 = \alpha p$, then $1 + \Sigma_3(3p+1) = \mathcal{M} p + r$, with $1 \leq r < p$.

Let $m = 3p+3$ and using the above statement, we have that

$$
\begin{aligned}
1 + \Sigma_3(m-1) &= 1 + \Sigma_3(3p+1) + (3p+2)!!! \\
&= \mathcal{M} \cdot p + r + 2 \cdot 5 \cdots \alpha p \cdots (3p+2) \\
&= [\mathcal{M} + 2 \cdot 5 \cdots \alpha \cdots (3p+2)] p + r = \mathcal{M} \cdot p + r\ ,
\end{aligned}
$$

because exists k, according to Theorem 2.78, $k \in \{0, 1, \ldots, p-1\}$, for which $3k + 2 = \alpha p$, then $1 + \Sigma_3(3p+2) = \mathcal{M} p + r$, with $1 \leq r < p$.

Through complete induction, it follows that $p \nmid \Sigma_3(m-1)$, for all $m > 3p$. □

With commands $k := 1..25$ and $SK(3, prime_k) \rightarrow$ (using the program 2.74) are obtained first 25 values of the function SK_3:

p	2	3	5	7	11	13	17	19	23	29	31	37
$SK_3(p)$	2	6	−1	4	5	7	22	11	61	70	11	55

41	43	47	53	59	61	67	71	73	79	83	89	97
80	32	29	154	24	145	8	98	21	30	24	22	90

. (2.97)

If $SK_3(p) = -1$, then for p function SK_3 is undefined.

2.25 Smarandache–Wagstaff Functions

2.25.1 Smarandache–Wagstaff Function of Order 1

Definition 2.80 ([Mudge, 1996a,b, Ashbacher, 1997]). The function $SW_1 : \mathbb{P}_{\geq 2} \to \mathbb{N}^*$, $m = SW_1(p)$ is the smallest $m \in \mathbb{N}^*$ such that $p \mid \Sigma_1(m)$, where $\Sigma_1(m)$ is defined by (2.92).

Proposition 2.81. *If $p \nmid \Sigma_1(m)$, for all $m < p$, then p never divides any sum for all $m \in \mathbb{N}^*$.*

Proof. If for all m, $m < p$, $p \nmid \Sigma_1(m)$, then $p \nmid \Sigma_1(p-1)$ i.e. $\Sigma_1(p-1) = \mathcal{M} \cdot p + r$, with $1 \leq r < p$. Let $m \geq p$,

$$\begin{aligned}\Sigma_1(m) &= \Sigma_1(p-1) + p! + (p+1)! + \ldots + m! \\ &= \mathcal{M} \cdot p + r + (p-1)![1 + (p+1) + \ldots + (p+1)(p+2)\cdots m]p \\ &= [\mathcal{M} + (p-1)!\big(1 + (p+1) + \ldots + (p+1)(p+2)\cdots m\big)]p + r \\ &= \mathcal{M} \cdot p + r\ ,\end{aligned}$$

then one obtains that $p \nmid \Sigma(m)$, for all m, $m \geq p$. □

Program 2.82. for calculating the values of functions SW_1, SW_2 and SW_3.

```
SW(k, p) := | for m ∈ 2..k·p − 1
            |    return m  if   mod (Σ(k, m), p)=0
            | return − 1
```

The program uses the subprograms Σ, 2.71, and the utilitarian function Mathcad *mod*.

With commands $k := 1..25$ and $SW(1, prime_k) \to$ (using the program 2.82) are obtained first 25 values of the function SW_1, [Weisstein, 2015e]:

p	2	3	5	7	11	13	17	19	23	29	31	37
$SW_1(p)$	−1	2	−1	−1	4	−1	5	−1	12	19	−1	24

41	43	47	53	59	61	67	71	73	79	83	89	97
32	19	−1	20	−1	−1	20	−1	7	57	−1	−1	6

(2.98)

If $SW_1(p) = -1$, then for p function SW_1 is undefined.

2.25.2 Smarandache–Wagstaff Function of Order 2

Definition 2.83. The function $SW_2 : \mathbb{P}_{\geq 2} \to \mathbb{N}^*$, $m = SW_2(p)$ is the smallest $m \in \mathbb{N}^*$ such that $p \mid \Sigma_2(m)$, where $\Sigma_2(m)$ is defined by (2.94).

Proposition 2.84. *If $p \nmid \Sigma_2(m)$, for all $m < 2p$, then p never divides any sum for all $m \in \mathbb{N}^*$.*

Proof. If for all m, $m < 2p$, $p \nmid \Sigma_2(m)$, then $p \nmid \Sigma_2(2p-1)$ i.e. $\Sigma_2(2p-1) = \mathscr{M} \cdot p + r$, with $1 \leq r < p$.

Let $m = 2p$,

$$\begin{aligned}\Sigma_2(m) &= \Sigma_2(2p-1) + 2p!! \\ &= \mathscr{M} \cdot p + r + 2 \cdot 4 \cdots (p-1)(p+1) \cdots 2p \\ &= [\mathscr{M} + 2 \cdot 4 \cdots (p-1)(p+1) \cdots 2(p-1) \cdot 2]\, p + r \\ &= \mathscr{M} \cdot p + r\ ,\end{aligned}$$

then $\Sigma_2(2p) = \mathscr{M}\, p + r$, with $1 \leq r < p$.

Let $m = 2p + 1$ and using the above statement, we have that

$$\begin{aligned}\Sigma_2(m) &= \Sigma_2(2p) + (2p+1)!! \\ &= \mathscr{M} \cdot p + r + 1 \cdot 3 \cdots (p-2) \cdot p \cdot (p+2) \cdots (2p+1) \\ &= [\mathscr{M} + 1 \cdot 3 \cdots (p-2)(p+2) \cdots 2(p+1)]\, p + r \\ &= \mathscr{M} \cdot p + r\ ,\end{aligned}$$

then complete induction, it follows that $p \nmid \Sigma_2(m)$, for all m, $m \geq 2p$. □

With commands $k := 1..25$ and $SW(2, prime_k) \to$ (using the program 2.82) are obtained first 20 values of the function SW_2:

p	2	3	5	7	11	13	17	19	23	29	31	37
$SW_2(p)$	3	2	−1	4	6	7	−1	12	34	5	26	52

41	43	47	53	59	61	67	71	73	79	83	89	97
36	−1	43	23	88	−1	21	−1	−1	59	48	−1	67

. (2.99)

If $SW_2(p) = -1$, then for p function SW_2 is undefined.

2.25.3 Smarandache–Wagstaff Function of Order 3

Definition 2.85. The function $SW_3 : \mathbb{P}_{\geq 2} \to \mathbb{N}^*$, $m = SW_3(p)$ is the smallest $m \in \mathbb{N}^*$ such that $p \mid \Sigma_3(m)$, where $\Sigma_3(m)$ is defined by (2.96).

Proposition 2.86. *If $p \nmid \Sigma_3(m)$, for all $m < 3p$, then $3p$ never divides any sum for all $m \in \mathbb{N}^*$.*

Proof. If for all m, $m < 3p$, $p \nmid \Sigma_3(m)$, then $p \nmid \Sigma_3(3p-1)$ i.e. $\Sigma_3(3p-1) = \mathscr{M} \cdot p + r$, with $1 \le r < p$.

Let $m = 3p$,

$$\begin{aligned}\Sigma_3(m) = \Sigma_3(3p-1) + 3p!!! &= \mathscr{M} \cdot p + r + 3 \cdot 6 \cdots 3(p-1)3p \\ &= [\mathscr{M} + 3 \cdot 6 \cdots 3(p-1) \cdot 3]\, p + r = \mathscr{M} \cdot p + r\ ,\end{aligned}$$

then $\Sigma_3(3p) = \mathscr{M} p + r$, with $1 \le r < p$.

Let $m = 3p + 1$ and using the above statement, we have that

$$\begin{aligned}\Sigma_3(m) &= \Sigma_3(3p) + (3p+1)!!! \\ &= \mathscr{M} \cdot p + r + 1 \cdot 4 \cdots \alpha p \cdots (3p+1) \\ &= [\mathscr{M} + 1 \cdot 4 \cdots \alpha \cdots (3p+1)]\, p + r = \mathscr{M} \cdot p + r\ ,\end{aligned}$$

because exists k, according to Theorem 2.78, $k \in \{0, 1, \ldots, p-1\}$, for which $3k + 1 = \alpha p$, then $\Sigma_3(3p+1) = \mathscr{M} p + r$, with $1 \le r < p$.

Let $m = 3p + 2$ and using the above statement, we have that

$$\begin{aligned}\Sigma_3(m) &= \Sigma_3(3p) + (3p+2)!!! \\ &= \mathscr{M} \cdot p + r + 2 \cdot 5 \cdots \alpha p \cdots (3p+2) \\ &= [\mathscr{M} + 2 \cdot 5 \cdots \alpha \cdots (3p+2)]\, p + r = \mathscr{M} \cdot p + r\ ,\end{aligned}$$

because exists, according to Theorem 2.78, $k \in \{0, 1, \ldots, p-1\}$, for which $3k+2 = \alpha p$, then $\Sigma_3(3p+2) = \mathscr{M} p + r$, with $1 \le r < p$.

Through complete induction, it follows that $p \nmid \Sigma_3(m)$, for all m, $m \ge 3p$. □

With commands $k := 1..25$ and $SW(3, prime_k) \rightarrow$ (using the program 2.82) are obtained first 25 values of the function SW_3:

p	2	3	5	7	11	13	17	19	23	29	31	37
$SW_3(p)$	3	2	4	9	7	17	18	6	−1	14	18	−1

41	43	47	53	59	61	67	71	73	79	83	89	97
13	13	73	−1	40	49	37	55	8	73	−1	132	72

. (2.100)

If $SW_3(p) = -1$, then for p function SW_3 is undefined.

2.26 Smarandache Near to k–Primorial Functions

2.26.1 Smarandache Near to Primorial Function

Let be the function $SNtP : \mathbb{N}^* \to \mathbb{P}_{\geq 2} \cup \{1\}$.

Definition 2.87. The number $p = SNtP(n)$ is the smallest prime, $p \leq n$, such that $mod[p\#-1, n] = 0 \vee mod[p\#, n] = 0 \vee mod[p\#+1, n] = 0$, where $p\#$ is the *primorial* of p, given by 1.1 .

Ashbacher [1996] shows that $SNtP(n)$ only exists, [Weisstein, 2015i].

Program 2.88. for calculating the function $SNtkP$.

```
SNtkP(n,k) := | return 1  if  n=1
              | m ← 1
              | while  prime_m ≤ k·n
              |   | kp ← kP(prime_m, k)
              |   | return prime_m  if   mod (kp,n)=0
              |   | return prime_m  if   mod (kp−1,n)=0
              |   | return prime_m  if   mod (kp+1,n)=0
              | return  −1
```

For $n = 1, 2, \ldots, 45$ the first few values of $SNtP(n) = SNtkP(n,1)$ are: 1, 2, 2, -1, 3, 3, 3, -1, -1, 5, 7, -1, 13, 7, 5, 43, 17, 47, 7, 47, 7, 11, 23, 47, 47, 13, 43, 47, 5, 5, 5, 47, 11, 17, 7, 47, 23, 19, 13, 47, 41, 7, 43, 47, 47 . If $SNtkP(n,1) = -1$, then for n function $SNtkP$ is undefined.

For examples $SNtkP(4) = -1$ because $4 \nmid (2\#-1) = 1$, $4 \nmid 2\# = 2$, $4 \nmid (2\#+1) = 3$, $4 \nmid (3\#-1) = 5$, $4 \nmid 3\# = 6$, $4 \nmid (3\#+1) = 7$.

2.26.2 Smarandache Near to Double Primorial Function

Let be the function $SNtDP : \mathbb{N}^* \to \mathbb{P}_{\geq 2} \cup \{1\}$.

Definition 2.89. The number $p = SNtDP(n)$ is the smallest prime, $p \leq 2n$, such that $\mod(p\#\#-1, n) = 0 \vee \mod(p\#\#, n) = 0 \vee \mod(p\#\#+1, n) = 0$, where $p\#\#$ is the *double primorial* of p, given by 1.3 .

For $n = 1, 2, \ldots, 45$ the first few values of $SNtDP(n) = SNtkP(n,2)$, 2.88, are: 2, 2, 2, 3, 5, -1, 7, 13, 5, 5, 5, 83, 13, 83, 83, 13, 13, 83, 19, 7, 7, 7, 23, 83, 37, 83, 23, 83, 29, 83, 31, 83, 89, 13, 83, 83, 11, 97, 13, 71, 23, 83, 43, 89, 89 . If $SNtkP(n,2) = -1$, then for n function $SNtkP$ is undefined.

2.26.3 Smarandache Near to Triple Primorial Function

Let be the function $SNtTP: \mathbb{N}^* \to \mathbb{P}_{\geq 2} \cup \{1\}$.

Definition 2.90. The number $p = SNtTP(n)$ is the smallest prime, $p \leq 3n$, such that $\mod(p\#\#\#-1, n) = 0 \vee \mod(p\#\#\#, n) = 0 \vee \mod(p\#\#\#+1, n) = 0$, where $p\#\#\#$ is the *triple primorial* of p, given by 1.4 .

For $n = 1, 2, \ldots, 40$ the first few values of $SNtTP(n) = SNtkP(n,3)$, given by 2.88, are: 2, 2, 2, 3, 5, 5, 7, 11, 23, 43, 11, 89, 7, 7, 7, 11, 11, 23, 19, 71, 37, 13, 23, 89, 71, 127, 97, 59, 29, 127, 31, 11, 11, 11, 127, 113, 37, 103, 29, 131, 41, 37, 31, 23, 131 . If $SNtkP(n,3) = -1$, then for n function $SNtkP$ is undefined.

We can generalize further this function as Smarandache Near to k-Primordial Function by using

$$p\underbrace{\#\ldots\#}_{k\ times},$$

defined analogously to 1.4, instead of $p\#\#\#$. Alternatives to $SNtkP(n)$ can be the following: $p\#\ldots\#$, or $p\#\ldots\#\pm 1$, or $p\#\ldots\#\pm 2$, or $\ldots$ $p\#\ldots\#\pm s$ (where s is a positive odd integer is a multiple of n).

2.27 Smarandache Ceil Function

Let the function $S_k: \mathbb{N}^* \to \mathbb{N}^*$.

Definition 2.91. The number $m = S_k(n)$ is the smallest $m \in \mathbb{N}^*$ such that $n \mid m^k$.

This function has been treated in the works [Smarandache, 1993a, Begay, 1997, Smarandache, 1997, Weisstein, 2015f].

Program 2.92. for the function S_k.

$$Sk(n,k) := \left| \begin{array}{l} for\ m \in 1..n \\ \quad return\ m\ \ if\ \ \mod(m^k, n) = 0 \end{array} \right.$$

If $n := 1..100$ then:

$Sk(n,1) \to$ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100 [Sloane, 2014, A000027];

$Sk(n,2) \to$ 1, 2, 3, 2, 5, 6, 7, 4, 3, 10, 11, 6, 13, 14, 15, 4, 17, 6, 19, 10, 21, 22, 23, 12, 5, 26, 9, 14, 29, 30, 31, 8, 33, 34, 35, 6, 37, 38, 39, 20, 41, 42, 43, 22, 15, 46, 47, 12, 7, 10, 51, 26, 53, 18, 55, 28, 57, 58, 59, 30, 61, 62, 21, 8, 65, 66, 67, 34, 69, 70, 71, 12, 73, 74, 15, 38, 77, 78, 79, 20, 9, 82, 83, 42, 85, 86, 87, 44, 89, 30, 91, 46, 93, 94, 95, 24, 97, 14, 33, 10 [Sloane, 2014, A019554];

$Sk(n,3) \to$ 1, 2, 3, 2, 5, 6, 7, 2, 3, 10, 11, 6, 13, 14, 15, 4, 17, 6, 19, 10, 21, 22, 23, 6, 5, 26, 3, 14, 29, 30, 31, 4, 33, 34, 35, 6, 37, 38, 39, 10, 41, 42, 43, 22, 15, 46, 47, 12, 7, 10, 51, 26, 53, 6, 55, 14, 57, 58, 59, 30, 61, 62, 21, 4, 65, 66, 67, 34, 69, 70, 71, 6, 73, 74, 15, 38, 77, 78, 79, 20, 9, 82, 83, 42, 85, 86, 87, 22, 89, 30, 91, 46, 93, 94, 95, 12, 97, 14, 33, 10 [Sloane, 2014, A019555];

$Sk(n,4) \to$ 1, 2, 3, 2, 5, 6, 7, 2, 3, 10, 11, 6, 13, 14, 15, 2, 17, 6, 19, 10, 21, 22, 23, 6, 5, 26, 3, 14, 29, 30, 31, 4, 33, 34, 35, 6, 37, 38, 39, 10, 41, 42, 43, 22, 15, 46, 47, 6, 7, 10, 51, 26, 53, 6, 55, 14, 57, 58, 59, 30, 61, 62, 21, 4, 65, 66, 67, 34, 69, 70, 71, 6, 73, 74, 15, 38, 77, 78, 79, 10, 3, 82, 83, 42, 85, 86, 87, 22, 89, 30, 91, 46, 93, 94, 95, 12, 97, 14, 33, 10 [Sloane, 2014, A053166];

$Sk(n,5) \to$ 1, 2, 3, 2, 5, 6, 7, 2, 3, 10, 11, 6, 13, 14, 15, 2, 17, 6, 19, 10, 21, 22, 23, 6, 5, 26, 3, 14, 29, 30, 31, 2, 33, 34, 35, 6, 37, 38, 39, 10, 41, 42, 43, 22, 15, 46, 47, 6, 7, 10, 51, 26, 53, 6, 55, 14, 57, 58, 59, 30, 61, 62, 21, 4, 65, 66, 67, 34, 69, 70, 71, 6, 73, 74, 15, 38, 77, 78, 79, 10, 3, 82, 83, 42, 85, 86, 87, 22, 89, 30, 91, 46, 93, 94, 95, 6, 97, 14, 33, 10 [Sloane, 2014, A007947];

$Sk(n,6) \to$ 1, 2, 3, 2, 5, 6, 7, 2, 3, 10, 11, 6, 13, 14, 15, 2, 17, 6, 19, 10, 21, 22, 23, 6, 5, 26, 3, 14, 29, 30, 31, 2, 33, 34, 35, 6, 37, 38, 39, 10, 41, 42, 43, 22, 15, 46, 47, 6, 7, 10, 51, 26, 53, 6, 55, 14, 57, 58, 59, 30, 61, 62, 21, 2, 65, 66, 67, 34, 69, 70, 71, 6, 73, 74, 15, 38, 77, 78, 79, 10, 3, 82, 83, 42, 85, 86, 87, 22, 89, 30, 91, 46, 93, 94, 95, 6, 97, 14, 33, 10;

2.28 Smarandache–Mersenne Functions

2.28.1 Smarandache–Mersenne Left Function

Let the function $SML : 2\mathbb{N}+1 \to \mathbb{N}^*$, where $2\mathbb{N}+1 = \{1,3,\ldots\}$ is the set of natural numbers odd.

Definition 2.93. The number $m = SML(\omega)$ is the smallest $m \in \mathbb{N}^*$ such that $\omega \mid 2^m - 1$.

Program 2.94. for generating the values of function *SML*.

$$SML(\omega) := \left| \begin{array}{l} for\ \ m = 1..\omega \\ \quad return\ \ m\ \ if\quad \bmod(2^m - 1, \omega) = 0 \\ return\ \ -1 \end{array} \right.$$

If $SML(\omega) = -1$, then for ω the function SML is undefined.

If $n := 1..40$ then:

$SML(prime_n) \to$ −1, 2, 4, 3, 10, 12, 8, 18, 11, 28, 5, 36, 20, 14, 23, 52, 58, 60, 66, 35, 9, 39, 82, 11, 48, 100, 51, 106, 36, 28, 7, 130, 68, 138, 148, 15, 52, 162, 83, 172,

$SML(2n-1) \to$: 1, 2, 4, 3, 6, 10, 12, 4, 8, 18, 6, 11, 20, 18, 28, 5, 10, 12, 36, 12, 20, 14, 12, 23, 21, 8, 52, 20, 18, 58, 60, 6, 12, 66, 22, 35, 9, 20, 30, 39 .

2.28.2 Smarandache–Mersenne Right Function

Let the function $SMR : 2\mathbb{N}+1 \to \mathbb{N}^*$, where $2\mathbb{N}+1 = \{1,3,\ldots\}$ is the set of natural numbers odd.

Definition 2.95. The number $m = SMR(\omega)$ is the smallest $m \in \mathbb{N}^*$ such that $\omega \mid 2^m + 1$.

Program 2.96. for generating the values of function *SMR*.

$$SMR(\omega) := \left| \begin{array}{l} for\ \ m = 1..\omega \\ \quad return\ m\ if\quad \bmod(2^m + 1, \omega) = 0 \\ return\ -1 \end{array} \right.$$

If $SMR(\omega) = -1$, then for ω the function SMR is undefined.

If $n := 1..40$ then:

$SMR(prime_n) \to$ −1, 1, 2, −1, 5, 6, 4, 9, −1, 14, −1, 18, 10, 7, −1, 26, 29, 30, 33, −1, −1, −1, 41, −1, 24, 50, −1, 53, 18, 14, −1, 65, 34, 69, 74, −1, 26, 81, −1, 86,

$SMR(2n-1) \to$ 1, 1, 2, −1, 3, 5, 6, −1, 4, 9, −1, −1, 10, 9, 14, −1, 5, −1, 18, −1, 10, 7, −1, −1, −1, −1, 26, −1, 9, 29, 30, −1, 6, 33, −1, −1, −1, −1, −1, -1 .

2.29 Smarandache–X-nacci Functions

2.29.1 Smarandache–Fibonacci Function

Let the function $SF : \mathbb{N}^* \to \mathbb{N}^*$ and Fibonacci sequence defined by formula $f_1 := 1$, $f_2 := 1$, $k = 1, 2, \ldots, 120$, $f_{k+2} := f_{k+1} + f_k$.

Definition 2.97. The number $m = SF(n)$ is the smallest $m \in \mathbb{N}^*$ such that $n \mid f_m$.

Program 2.98. for generating the values of function *SF*.

$$SF(n) := \left| \begin{array}{l} for \;\; m \in 1..last(f) \\ \quad return \;\; m \;\; if \quad \mathrm{mod}\,(f_m, n) = 0 \\ return \;\; -1 \end{array} \right.$$

If $SF(n) = -1$, then for n the function *SF* is undefined for $last(f) = 120$.

If $n := 1..80$ then $SF(n)^{\mathrm{T}} \to$ 1, 3, 4, 6, 5, 12, 8, 6, 12, 15, 10, 12, 7, 24, 20, 12, 9, 12, 18, 30, 8, 30, 24, 12, 25, 21, 36, 24, 14, 60, 30, 24, 20, 9, 40, 12, 19, 18, 28, 30, 20, 24, 44, 30, 60, 24, 16, 12, 56, 75, 36, 42, 27, 36, 10, 24, 36, 42, 58, 60, 15, 30, 24, 48, 35, 60, 68, 18, 24, 120, 70, 12, 37, 57, 100, 18, 40, 84, 78, 60 .

2.29.2 Smarandache–Tribonacci Function

Let the function $STr : \mathbb{N}^* \to \mathbb{N}^*$ and Tribonacci sequence defined by formula $t_1 := 1$, $t_2 := 1$, $t_3 := 2$, $k = 1, 2, \ldots, 130$, $t_{k+3} := t_{k+2} + t_{k+1} + t_k$.

Definition 2.99. The number $m = STr(n)$ is the smallest $m \in \mathbb{N}^*$ such that $n \mid t_m$.

Program 2.100. for generating the values of function *STr*.

$$STr(n) := \left| \begin{array}{l} for \;\; m = 1..last(t) \\ \quad return \;\; m \;\; if \quad \mathrm{mod}\,(t_m, n) = 0 \\ return \;\; -1 \end{array} \right.$$

If $STr(n) = -1$, then for n the function *STr* is undefined for $last(t) = 100$.

If $n := 1..80$ then $STr(n)^{\mathrm{T}} \to$ 1, 3, 7, 4, 14, 7, 5, 7, 9, 19, 8, 7, 6, 12, 52, 15, 28, 12, 18, 31, 12, 8, 29, 7, 30, 39, 9, 12, 77, 52, 14, 15, 35, 28, 21, 12, 19, 28, 39, 31, 35, 12, 82, 8, 52, 55, 29, 64, 15, 52, 124, 39, 33, 35, 14, 12, 103, 123, 64, 52, 68, 60, 12, 15, 52, 35, 100, 28, 117, 31, 132, 12, 31, 19, 52, 28, 37, 39, 18, 31 .

2.29.3 Smarandache–Tetranacci Function

Let the function $STe\colon \mathbb{N}^* \to \mathbb{N}^*$ and Tetranacci sequence defined by formula $T_1 := 1$, $T_2 := 1$, $T_3 := 2$, $T_4 := 4$, $k = 1, 2, \ldots, 300$, $T_{k+4} := T_{k+3} + T_{k+2} + T_{k+1} + T_k$.

Definition 2.101. The number $m = STe(n)$ is the smallest $m \in \mathbb{N}^*$ such that $n \mid T_m$.

Program 2.102. for generating the values of function STe.

$$STe(n) := \left| \begin{array}{l} for\ m = 1..last(T) \\ \quad return\ m\ if\ \ \mod(T_m, n) = 0 \\ return\ -1 \end{array} \right.$$

If $STe(n) = -1$, then for n the function STe is undefined for $last(T) = 300$.

If $n := 1..80$ then $STe(n)^{\mathrm{T}} \to$ 1, 3, 6, 4, 6, 9, 8, 5, 9, 13, 20, 9, 10, 8, 6, 10, 53, 9, 48, 28, 18, 20, 35, 18, 76, 10, 9, 8, 7, 68, 20, 15, 20, 53, 30, 9, 58, 48, 78, 28, 19, 18, 63, 20, 68, 35, 28, 18, 46, 108, 76, 10, 158, 9, 52, 8, 87, 133, 18, 68, 51, 20, 46, 35, 78, 20, 17, 138, 35, 30, 230, 20, 72, 58, 76, 48, 118, 78, 303, 30.

And so on, one can define the Smarandache–N-nacci function, where N-nacci sequence is 1, 1, 2, 4, 8, ... and $N_{n+k} = N_{n+k-1} + N_{n+k-2} + \ldots + N_n$ is the sum of the previous n terms. Then, the number $m = SN(n)$ is the smallest m such that $n \mid N_m$.

2.30 Pseudo–Smarandache Functions

The functions in this section are similar to Smarandache S function, 2.67, [Smarandache, 1980, Cira and Smarandache, 2014]. The first authors who dealt with the definition and properties of the pseudo–Smarandache function of first rank are: Ashbacher [1995] and Kashihara [1996], [Weisstein, 2015d].

2.30.1 Pseudo–Smarandache Function of the Order 1

Let n be a natural positive number and function $Z_1 : \mathbb{N}^* \to \mathbb{N}^*$.

Definition 2.103. The value $Z_1(n)$ is the smallest natural number $m = Z_1(n)$ for which the sum $1 + 2 + \ldots + m$ divides by n.

Considering that $1 + 2 + \ldots + m = m(m+1)/2$ this definition of the function Z_1 is equivalent with the fact that $m = Z_1(n)$ is the smallest natural number n for which we have $m(m+1) = \mathscr{M} \cdot 2n$ i.e. $m(m+1)$ is multiple of $2n$ $\big($or the equivalent relation $2n \mid m(m+1)$ i.e. $2n$ divides $m(m+1)\big)$.

Lemma 2.104. *Let $n, m \in \mathbb{N}^*$, $n \geq m$, if $n \mid [m(m+1)]/2$, then $m \geq \lceil s_1(n) \rceil$, where*

$$s_1(n) := \frac{\sqrt{8n+1}-1}{2} \ . \tag{2.101}$$

Proof. The relation $n \mid [m(m+1)]/2$ is equivalent with $m(m+1) = \mathscr{M} \cdot 2n$, with $\mathscr{M} = 1, 2, \ldots$. The smallest multiplicity is for $\mathscr{M} = 1$. The equation $m(m+1) = 2n$ has as positive real solution $s_1(n)$ given by (2.101). Considering that m is a natural number, it follows that $m \geq \lceil s_1(n) \rceil$. □

The bound of the function Z_1 (see Figure 2.13) is given by the theorem:

Theorem 2.105. *For any $n \in \mathbb{N}^*$ we have $\lceil s_1(n) \rceil \leq Z_1(n) \leq 2n-1$.*

Proof. T he inequality $\lceil s_1(n) \rceil \leq Z_1(n)$, for any $n \in \mathbb{N}^*$ follows from Lemma 2.104.

The relation $n \mid [m(m+1)]/2$ is equivalent with $2n \mid [m(m+1)]$. Of the two factors of expression $m(m+1)$, in a sequential ascending scroll, first with value $2n$ is $m+1$. It follows that for $m = 2n-1$ we first met the condition $n \mid [m(m+1)/2]$, i.e. $m(m+1)/2n = (2n-1)2n/2n = 2n-1 \in \mathbb{N}^*$. □

Theorem 2.106. *For any $k \in \mathbb{N}^*$, it follows that $Z_1(2^k) = 2^{k+1}-1$.*

Proof. We use the notation $Z_1(n) = m$. If $n = 2^k$, we calculate $m(m+1)/(2n)$ for $m = 2^{k+1}-1$.

$$\frac{m(m+1)}{2} = \frac{(2^{k+1}-1)2^{k+1}}{2 \cdot 2^k} = 2^{k+1}-1 \in \mathbb{N}^* \ .$$

Let us prove that $m = 2^{k+1}-1$ is the smallest m for which $m(m+1)/(2n) \in \mathbb{N}^*$. It is obvious that m has to be of form $2^\alpha - 1$ sau 2^α, where $\alpha \in \mathbb{N}^*$, if we want $m(m+1)$ to divide by $2 \cdot 2^k$. Let $m = 2^{k+1-j}-1$ with $j \in \mathbb{N}^*$, which is a number smaller than $2^{k+1}-1$. If we calculate

$$\frac{m(m+1)}{2n} = \frac{(2^{k+1-j}-1)2^{k+1-j}}{2 \cdot 2^k} = (2^{k+1-j}-1)2^{-j} \notin \mathbb{N}^* \ ,$$

therefore we can not have $Z_1(2^k) = m = 2^{k+1-j}-1$, with $j \in \mathbb{N}^*$. Let $m = 2^{k+1-j}$, with $j \in \mathbb{N}^*$, which is a number smaller than $2^{k+1}-1$. Calculating,

$$\frac{m(m+1)}{2n} = \frac{2^{k+1-j}(2^{k+1-j}+1)}{2 \cdot 2^k} = 2^{-j}(2^{k+1-j}+1) \notin \mathbb{N}^* \ ,$$

therefore we can not have $Z_1(2^k) = m = 2^{k+1-j}$, with $j \in \mathbb{N}^*$.

It was proved that $m = 2^{k+1}-1$ is the smallest number that has the property $n \mid m(m+1)/2$, therefore $Z_1(2^k) = 2^{k+1}-1$. □

We present a theorem, [Kashihara, 1996, T4, p. 36], on function values Z_1 for the powers of primes.

Theorem 2.107. $Z_1(p^k) = p^k - 1$ *for any* $p \in \mathbb{P}_{\geq 3}$ *and* $k \in \mathbb{N}^*$.

Proof. From the sequential ascending completion of m,the first factor between m and $m+1$, which divides $n = p^k$ is $m+1 = p^k$. Then it follows that $m = p^k - 1$. It can be proved by direct calculation that $m(m+1)/(2n) = (p^k-1)p^k/(2p^k) \in \mathbb{N}^*$ because $p^k - 1$, since $p \in \mathbb{P}_{\geq 3}$, is always an even number. Therefore, $m = p^k - 1$ is the smallest natural number for which $m(m+1)/2$ divides to $n = p^k$, then it follows that $Z_1(p^k) = p^k - 1$ for any $p \in \mathbb{P}_{\geq 3}$. □

Corollary 2.108. [Kashihara, 1996, T3, p. 36] *For* $k = 1$ *it follows that* $Z_1(p) = p - 1$ *for any* $p \in \mathbb{P}_{\geq 3}$.

Program 2.109. for the function Z_1.

$$
Z_1(n) := \left|
\begin{array}{l}
\mathit{return}\ n-1\ \ \mathit{if}\ \ TS(n){=}1 \wedge n > 2 \\
\mathit{for}\ \ m \in \mathit{ceil}(s_1(n))..n-1 \\
\quad \mathit{return}\ m\ \mathit{if}\ \mathit{mod}[m(m+1), 2n]{=}0 \\
\mathit{return}\ 2n-1
\end{array}
\right.
$$

Explanations for the program Z_1, for search of m optimization.

1. The program Z_1 uses Smarandache primality test TS, 1.5. For $n \in \mathbb{P}_{\geq 3}$ the value of function Z_1 is $n-1$ according the Corollary 2.108, without "searching" m anymore, the value of the function Z_1, that fulfills the condition $n \mid [m(m+1)/2]$.

2. Searching for m in program Z_1 starts from $\lceil s_1(n) \rceil$ according to the Theorem 2.105. Searching for m ends when m has at most value $n-1$.

3. If the search for m reached the value of $n-1$ and the condition $mod[m(m+1)/2, n] = 0$ (i.e. the rest of division $m(m+1)/2$ to n is 0) was fulfilled, then it follows that n is of form 2^k. Indeed, if $m = n-1$ and
$$\frac{(n-1)(n-1+1)}{2n} = \frac{n-1}{2} \notin \mathbb{N}^* ,$$
then it follows that n is an even number, and that is $n = 2q_1$. Calculating again
$$\frac{(2q_1-1)(2q_1-1+1)}{2 \cdot 2q_1} = \frac{2q_1-1}{2} \notin \mathbb{N}^* ,$$

then it follows that q_1 is even, i.e. $q_1 = 2q_2$ and $n = 2 \cdot 2q_2 = 2^2 \cdot q_2$. After an identical reasoning, it follows that $n = 2^3 \cdot q_3$, and so on. But n is a finite number, then it follows that there exists k $k \in \mathbb{N}^*$ such that $n = 2^k$. Therefore, according to the Theorem 2.106 we have that $Z_1(n) = Z_1(2^k) = 2^{k+1} - 1 = 2n - 1$, see Figure 2.13.

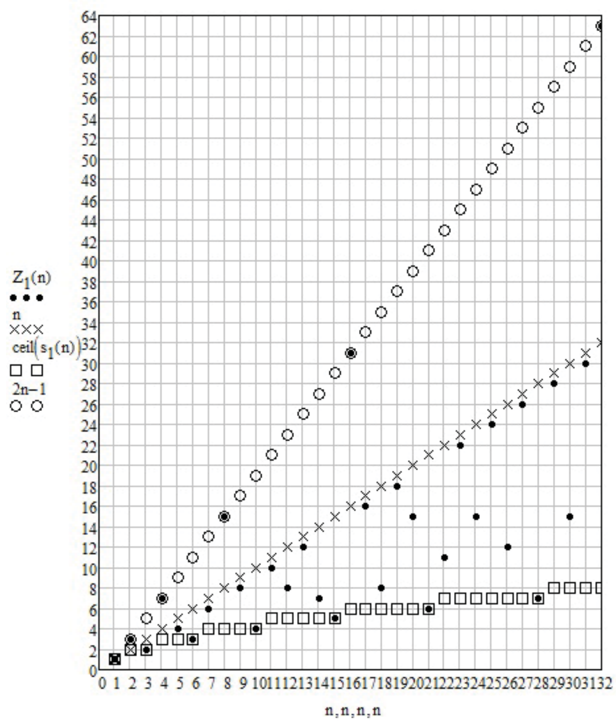

Figure 2.13: Function Z_1

2.30.2 Pseudo–Smarandache Function of the Order 2

We define the function $Z_2 : \mathbb{N}^* \to \mathbb{N}^*$ and denote the value of the function Z_2 with m, i.e. $m = Z_2(n)$. The value of m is the smallest natural number for which the sum $1^2 + 2^2 + \ldots + m^2$ divides by n.

Considering that $1^2 + 2^2 + \ldots + m^2 = m(m+1)(2m+1)/6$ this definition of the function Z_2 is equivalent with $m = Z_2(n)$ is the smallest natural number for which we have $m(m+1)(2m+1)/6 = \mathscr{M} \cdot n$ i.e. $m(m+1)(2m+1)$ is multiple of $6n$ $\big($or the equivalent relation $6n \mid m(m+1)(2m+1)$ i.e. $6n$ divides $m(m+1)(2m+1)\big)$.

We consider the function τ given by the formula:

$$\tau(n) := \sqrt[3]{3(108n + \sqrt{11664n^2 - 3})}\,, \tag{2.102}$$

the real solution of the equation $m(m+1)(2m+1) = 6n$ is

$$s_2(n) := \frac{1}{2}\left(\frac{1}{\tau(n)} + \frac{\tau(n)}{3} - 1\right). \tag{2.103}$$

Lemma 2.110. *Let $n, m \in \mathbb{N}^*$, $n \geq m$, if $n \mid [m(m+1)(2m+1)]/6$, then $m \geq \lceil s_2(n) \rceil$, with $s_2(n)$ is given by* (2.103).

Proof. The relation $n \mid [m(m+1)(2m+1)]/6 \Leftrightarrow m(m+1)(2m+1) = \mathscr{M} \cdot 6n$, with $\mathscr{M} = 1, 2, \ldots$. The smallest multiplicity is for $\mathscr{M} = 1$. The equation $m(m+1)(2m+1) = 6n$ has as real positive solution $s_2(n)$ given by (2.103). Considering that m is a natural number, it follows that $m \geq \lceil s_2(n) \rceil$. $\square$

Lemma 2.111. *The number $(2^{k+2} - 1)(2^{k+1} - 1)$ is multiple of 3 for any $k \in \mathbb{N}^*$.*

Proof. Let us observe that for:

- $k = 1$, $2^{k+1} - 1 = 2^2 - 1 = 3$ and $2^{k+2} - 1 = 2^3 - 1 = 7$,
- $k = 2$, $2^{k+1} - 1 = 2^3 - 1 = 7$ and $2^{k+2} - 1 = 2^4 - 1 = 15$,
- $\ldots$,
- $k = 2j - 1$, $2^{2j} - 1 = 3 \cdot \mathscr{M}$ and $2^{2j+1} - 1 = ?$,
- $k = 2j$, $2^{2j+1} - 1 = ?$ and $2^{2(j+1)} - 1 = 3 \cdot \mathscr{M}$,
- and so on.

We can say that the proof of lemma is equivalent with proofing the fact that $2^{2j} - 1$ is multiple of 3 for anye $j \in \mathbb{N}^*$.

We make the proof by full induction.

- For $j = 1$ we have $2^2 - 1 = 3$, is multiple of 3.
- We suppose that $2^{2j} - 1 = 3 \cdot \mathscr{M}$.
- Then we show that $2^{2(j+1)} - 1 = 3 \cdot \mathscr{M}$. Indeed

$$\begin{aligned} 2^{2(j+1)} - 1 = 2^2 \cdot 2^{2j} - 1 = 2^2(\cdot 2^{2j} - 1) + 2^2 - 1 = 2^2 \cdot 3\mathscr{M} + 3 \\ = 3 \cdot (2^2 \mathscr{M} + 1) = 3 \cdot \mathscr{M}\,. \end{aligned}$$

□

If $k = 1, 2, \ldots, 10$, then

$$(2^{k+2}-1)(2^{k+1}-1) = \begin{pmatrix} 3\cdot 7 \\ 3\cdot 5\cdot 7 \\ 3\cdot 5\cdot 31 \\ 3^2\cdot 7\cdot 31 \\ 3^2\cdot 7\cdot 127 \\ 3\cdot 5\cdot 17\cdot 127 \\ 3\cdot 5\cdot 7\cdot 17\cdot 73 \\ 3\cdot 7\cdot 11\cdot 31\cdot 73 \\ 3\cdot 11\cdot 23\cdot 31\cdot 89 \\ 3^2\cdot 5\cdot 7\cdot 13\cdot 23\cdot 89 \end{pmatrix} .$$

The bound of the function Z_2 (see Figure 2.14) is given by the theorem:

Theorem 2.112. *For any $n \in \mathbb{N}^*$ we have $\lceil s_2(n) \rceil \le Z_2(n) \le 2n-1$.*

Proof. The inequality $\lceil s_2(n) \rceil \le Z_2(n)$, for any $n \in \mathbb{N}^*$ results from Lemma 2.110. If $n = 2^k$ $m = 2^{k+1}-1$, then

$$\frac{m(m+1)(2m+1)}{6n} = \frac{(2^{k+1}-1)2^{k+1}(2^{k+2}-2+1)}{6\cdot 2^k} = \frac{(2^{k+1}-1)(2^{k+2}-1)}{3} ,$$

by according to Lemma 2.111 the number $(2^{k+1}-1)(2^{k+2}-1)$ is multiple of 3. Then it results that $6n \mid m(m+1)(2m+1)$, therefore we can say that $Z_2(n) = 2n-1$, if $n = 2^k$, for any $k \in \mathbb{N}^*$.

Let us prove that if $Z_2(n) = 2n-1$, then $n = 2^k$. If $Z_2(n) = 2n-1$, then it results that $6n \mid (2n-1)2n(4n-1)$, i.e. $(2n-1)(4n-1) = 3\cdot\mathcal{M}$. Let us suppose that n is of form $n = p^k$, where $p \in \mathbb{P}_{\ge 2}$ and $k \in \mathbb{N}^*$. We look for the pair $(p, \mathcal{M})$, of integer number, solution of the system:

$$\begin{cases} (2p-1)(4p-1) = 3\cdot\mathcal{M} , \\ (2p^2-1)(4p^2-1) = 3\cdot q\cdot\mathcal{M} \end{cases} \tag{2.104}$$

for $q = 1, 2, \ldots$. The first value of q for which we also have a pair $(p,\ \mathcal{M})$ of integer numbers as solution of the nonlinear system (2.104) is $q = 5$. The nonlinear system:

$$\begin{cases} (2p-1)(4p-1) = 3\cdot\mathcal{M} , \\ (2p^2-1)(4p^2-1) = 3\cdot 5\cdot\mathcal{M} , \end{cases}$$

has the solutions:

$$\begin{pmatrix} p & \mathscr{M} \end{pmatrix} = \begin{pmatrix} 2 & 7 \\ \frac{1}{2} & 0 \\ -\frac{\sqrt{33}}{4} - \frac{5}{4} & \frac{25}{2} + \frac{13\sqrt{33}}{6} \\ \frac{\sqrt{33}}{4} - \frac{5}{4} & \frac{25}{2} - \frac{13\sqrt{33}}{6} \end{pmatrix}$$

It follows that the first solution n for which $(2n-1)(4n-1)$ is always multiple of 3 is $n = 2^k$. As we have seen in Lemma 2.111 for any $k \in \mathbb{N}^*$, $(2^{k+1}-1)(2^{k+2}-1) = 3 \cdot \mathscr{M}$. □

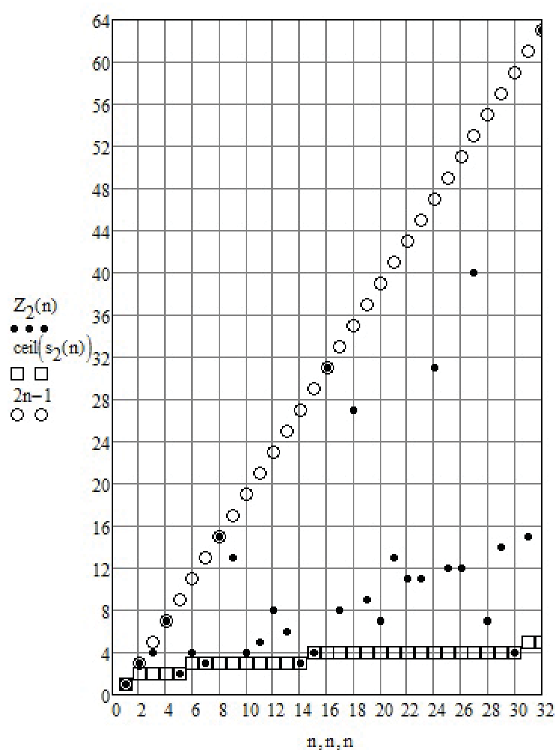

Figure 2.14: Function Z_2

Program 2.113. for function Z_2.

$$Z_2(n) := \left| \begin{array}{l} for\ m \in ceil(s_2(n))..2n-1 \\ \quad return\ m\ \ if\ \ mod[m(m+1)(2m+1), 6n]{=}0 \end{array} \right.$$

2.30.3 Pseudo–Smarandache Function of the Order 3

We define the function $Z_3 : \mathbb{N}^* \to \mathbb{N}^*$ and denote the value of the function Z_3 cu m, i.e. $m = Z_3(n)$. The value of m is the smallest natural number for which the sum $1^3 + 2^3 + \ldots + m^3$ is dividing by n.

Considering the fact that $1^3 + 2^3 + \ldots + m^3 = [m(m+1)/2]^2$ this definition of the function Z_3 is equivalent with the fact that $m = Z_3(n)$ is the smallest natural number for which we have $[m(m+1)/2]^2 = \mathscr{M} \cdot n$ i.e. $[m(m+1)]^2$ is multiple of $4n$ $\big($or the equivalent relation $4n \mid [m(m+1)]^2$ i.e. $4n$ divides $[m(m+1)]^2\big)$.

The function $s_3(n)$ is the real positive solution of the equation $m^2(m+1)^2 = 4n$.

$$s_3(n) := \frac{\sqrt{8\sqrt{n}+1}-1}{2}\ . \tag{2.105}$$

Lemma 2.114. *Let $n, m \in \mathbb{N}^*$, $n \geq m$, if $n \mid [m^2(m+1)^2]/4$, then $m \geq \lceil s_3(n) \rceil$, where $s_3(n)$ is given by* (2.105).

Proof. The relation $n \mid [m^2(m+1)^2]/4 \Leftrightarrow m^2(m+1)^2 = \mathscr{M} \cdot 4n$, with $\mathscr{M} = 1, 2, \ldots$. The smallest multiplicity is for $\mathscr{M} = 1$. The equation $m^2(m+1)^2 = 4n$ has as real positive solution $s_3(n)$ given by (2.105). Considering that m is a natural number, it results that $m \geq \lceil s_3(n) \rceil$. □

Theorem 2.115. *For any number $p \in \mathbb{P}_{\geq 3}$, $Z_3(p) = p - 1$.*

Proof. We use the notation $Z_3(n) = m$. If $p \in \mathbb{P}_{\geq 3}$ then $p = 2k+1$, i.e. p is an odd number, and $p - 1 = 2k$, i.e. $p - 1$ is an even number. Calculating for $n = p$ the ratio

$$\frac{(p-1)^2 p^2}{4p} = \frac{4k^2(2k+1)^2}{4(2k+1)} = k^2(2k+1) \in \mathbb{N}^*\ ,$$

it follows that, for $m = p - 1$, $n = p$ divides $m^2(m+1)^2/4 = 1^1 + 2^3 + \ldots + m^3$.

Let us prove that $m = p - 1$ is the smallest integer for which we have this property. Supposing that there is a $m = p - j$, where $j \geq 2$, such that $Z_3(p) = p - j$, then it should that the number $(p-j)^2(p-j+1)^2/4$ divides p, i.e. $p \mid (p-j)$ or $p \mid (p-j+1)$ which is absurd. Therefore, $m = p-1$ is the smallest m for which we have that $m^2(m+1)^2/4$ divides p. □

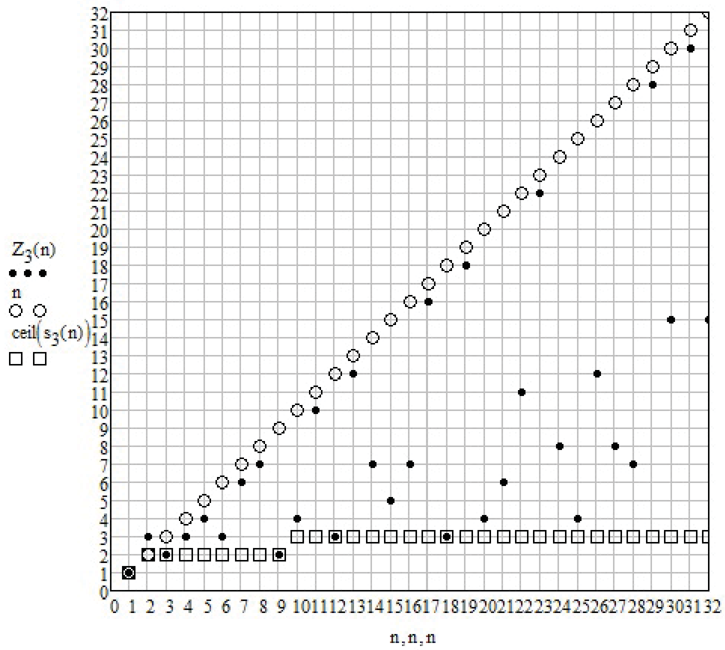

Figure 2.15: Function Z_3

Theorem 2.116. *For any $n \in \mathbb{N}^*$, $n \geq 3$, $Z_3(n) \leq n-1$.*

Proof. We use the notation $Z_3(n) = m$. Suppose that $Z_3(n) \geq n$. If $Z_3(n) = n$, then itshould that $4n \mid [n^2(n+1)^2]$, but

$$\frac{n^2(n+1)^2}{4n} = \frac{n(n+1)^2}{4} .$$

1. If $n = 2n_1$, then
$$\frac{n(n+1)^2}{4} = \frac{n_1(2n_1+1)^2}{2}$$

   (a) if $n_1 = 2n_2$, then
$$\frac{n(n+1)^2}{4} = \frac{n_1(2n_1+1)^2}{2} = \frac{2n_2(4n_2+1)^2}{2} = n_2(4n_2+1)^2 \in \mathbb{N}^* ,$$

   (b) if $n_1 = 2n_2 + 1$ then
$$\frac{n(n+1)^2}{4} = \frac{n_1(2n_1+1)^2}{2} = \frac{(2n_2+1)(4n_2+3)^2}{2} \notin \mathbb{N}^* ,$$

from where it results that the supposition $Z_3(n) = n$ is false (true only if $n = 4n_2$).

2. If $n = 2n_1 + 1$, then

$$\frac{n(n+1)^2}{4} = (2n_1+1)(n_1+1)^2 \in \mathbb{N}^*\ ,$$

but that would imply that also for primes, which are odd numbers, we would have $Z_3(p) = p$ which contradicts the Theorem 2.115, so the supposition that $Z_3(n) = n$ is false.

In conclusion, the supposition that $Z_3(n) = n$ is false. Similarly, one can prove that $Z_3(n) = n + j$, for $j = 1, 2, \ldots$, is false. Therefore, it follows that the equality $Z_3(n) \le n - 1$ is true. □

Observation 2.117. We have two exceptional cases $Z_3(1) = 1$ and $Z_3(2) = 3$.

Theorem 2.118. *For any $n \in \mathbb{N}^*$, $n \ge 3$ and $n \notin \mathbb{P}_{\ge 3}$, we have $Z_3(n) \le \left\lfloor \frac{n}{2} \right\rfloor$.*

Proof. Theorem to be proved! □

Theorem 2.119. *For any $k \in \mathbb{N}^*$, $Z_3(2^k) = 2^{\lceil \frac{k+2}{2} \rceil} - 1$.*

Proof. We use the notation $Z_3(n) = m$. If $n = 2^k$,then, by direct calculation, it verifies for $m = 2^{\lceil \frac{k+2}{2} \rceil} - 1$, $m^2(m+1)^2$ divides by $4n$,

$$\frac{\left(2^{\left\lceil \frac{k+2}{2} \right\rceil} - 1\right)^2 \left(2^{\left\lceil \frac{k+2}{2} \right\rceil}\right)^2}{4 \cdot 2^k} = \frac{\left(2^{\left\lceil \frac{k+2}{2} \right\rceil} - 1\right)^2 2^{k+2}}{2^{k+2}} = \left(2^{\left\lceil \frac{k+2}{2} \right\rceil} - 1\right)^2 \in \mathbb{N}^*\ .$$

Let us prove that $m = 2^{\lceil \frac{k+2}{2} \rceil} - 1$ is the smallest natural number for which $m^2(m+1)^2$ divides by $4n$. We search for numbers m of the forma $p^k - 1$. From the divisibility conditions for $k = 2$ and $k = 4$, it follows the nonlinear system

$$\begin{cases} (p^2-1)p^2 = 2^2 \cdot \mathcal{M}\ , \\ (p^4-1)p^4 = 2^3 \cdot q \cdot \mathcal{M}\ , \end{cases} \tag{2.106}$$

for $q = 1, 2, \ldots$. The number $q = 10$ is the first natural number for which the system (2.106) has integer positive solution. We present the solution of the system using Mathcad symbolic computation

$$\begin{bmatrix} (p^2-1)p^2 = 2^2 \cdot \mathcal{M}\ , \\ (p^4-1)p^4 = 2^3 \cdot 10 \cdot \mathcal{M}\ , \end{bmatrix} \left| \begin{array}{l} assume, p{=}integer \\ assume, \mathcal{M}{=}integer \\ solve, \begin{pmatrix} p \\ \mathcal{M} \end{pmatrix} \end{array} \right. \to \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ -1 & 0 \\ 2 & 3 \\ -2 & 3 \end{pmatrix}.$$

Of the 5 solutions only one solution is convenient $p=2$ and $\mathscr{M}=3$.It follows that $m=2^{f(k)}-1$. By direct verification it follows that $f(k)=\lceil\frac{k+2}{2}\rceil$. Therefore $m=2^{\lceil\frac{k+2}{2}\rceil}-1$ is the smallest natural number for which $m^2(m+1)^2$ divides by $4n$. □

Program 2.120. for function Z_3.

$$
Z_3(n) := \left|
\begin{array}{l}
\mathit{return}\ 3\ \ \mathit{if}\ \ n{=}2\\
\mathit{return}\ n-1\ \mathit{if}\ n>2\wedge TS(n){=}1\\
\mathit{for}\ \ m\in \mathit{ceil}(s_3(n))..n\\
\quad \mathit{return}\ m\ \mathit{if}\ \mathit{mod}\left[[m(m+1)]^2, 4n\right]{=}0
\end{array}\right.
$$

Explanations for the program Z_3, 2.120, for search of m shortening m.

1. The program treats separately the exceptional case $Z_3(2)=3$.

2. The search of m begins from the value $\lceil s_3(n)\rceil$ according to the Lemma 2.114.

3. The search of m goes to the value $m=n$.

4. The program uses the Smarandache primality test, 1.5. If $TS(n)=1$, then $n\in\mathbb{P}_{\geq 3}$ and $Z_3(n)=n-1$ according to the Theorem 2.115.

2.30.4 Alternative Pseudo–Smarandache Function

We can define alternatives of the function Z_k, $k=1,2,3$. For example: $V_k:\mathbb{N}^*\to\mathbb{N}^*$, $m=V(n)$ is the smallest integer m such that

$$n \mid 1^k-2^k+3^k-4^k+\ldots+(-1)^{m-1}m^k\ .$$

To note that:

$$1-2+3-4+\ldots+(-1)^{m-1}\cdot m=\frac{(-1)^{m+1}(2m+1)+1}{4}\ ,$$

$$1^2-2^2+3^2-4^2+\ldots+(-1)^{m-1}\cdot m^2=\frac{(-1)^{m+1}m(m+1)}{2}\ ,$$

and

$$1^3-2^3+3^3-4^3+\ldots+(-1)^{m-1}\cdot m^3 = \frac{(-1)^{m+1}(2m+1)(2m^2+2m-1)-1}{8}\ .$$

Or more versions of Z_k, $k=1,2,3$, by inserting in between the numbers 1, 2, 3, … various operators.

2.30.5 General Smarandache Functions

Function $T_k : \mathbb{N}^* \to \mathbb{N}^*$, $m = T_k(n)$ is smallest integer m such that $1^k \circ 2^k \circ \ldots \circ m^k$ is divisible by m, where $\circ \in \left\{+, \cdot, (-1)^{i-1}\right\}$ (and more operators can be used).

If $\circ \equiv \cdot$ we have Smarandache's functions, if $\circ \equiv +$ then result Pseudo–Smarandache functions and if $\circ \equiv (-1)^{i-1}$ then one obtains Alternative–Smarandache functions.

2.31 Smarandache Functions of the k-th Kind

2.31.1 Smarandache Function of the First Kind

Let the function $S_n : \mathbb{N}^* \to \mathbb{N}^*$ with $n \in \mathbb{N}^*$.

Definition 2.121.

1. If $n = p^\alpha$, where $p \in \mathbb{P}_{\geq 2} \cup \{1\}$ and $\alpha \in \mathbb{N}^*$, then $m = S_n(a)$ is smallest positive integer such that $n^a \mid m!$;

2. If $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_s^{\alpha_s}$, where $p_j \in \mathbb{P}_{\geq 2}$ and $\alpha_j \in \mathbb{N}^*$ for $j = 1, 2, \ldots s$, then

$$S_n(a) = \max_{1 \leq j \leq s} \left\{ S_{p_j^{\alpha_j}}(a) \right\} .$$

2.31.2 Smarandache Function of the Second Kind

Smarandache functions of the second kind: $S^k : \mathbb{N}^* \to \mathbb{N}^*$, $S^k(n) = S_n(k)$ for $k \in \mathbb{N}^*$ where S_n are the Smarandache functions of the first kind.

2.31.3 Smarandache Function of the Third Kind

Smarandache function of the third kind: $S_a^b(n) = S_{a_n}(b_n)$, where S_{a_n} is the Smarandache function of the first kind, and the sequences $\{a_n\}$ and $\{b_n\}$ are different from the following situations:

1. $a_n = 1$ and $b_n = n$, for $n \in \mathbb{N}^*$;

2. $a_n = n$ and $b_n = 1$, for $n \in \mathbb{N}^*$.

2.32 The Generalization of the Factorial

2.32.1 Factorial for Real Numbers

Let $x \in \mathbb{R}_+$, be positive real number. Then factorial of real number is defined as, [Smarandache, 1972]:

$$x! = \prod_{k=0}^{\lfloor x \rfloor} (x-k) \ , \ \text{ where } \ k \in \mathbb{N} \ . \tag{2.107}$$

Examples:

1. $2.5! = 2.5(2.5-1)(2.5-2) = 1.875$,
2. $4.37! = 4.37(4.37-1)(4.37-2)(4.37-3)(4.37-4) = 17.6922054957$.

More generally.

Let $\delta \in \mathbb{R}_+$ be positive real number, then we can introduce formula:

$$x!(\delta) = \prod_{k=0}^{k\cdot\delta<x} (x-k\cdot\delta) \ , \ \text{ where } \ k \in \mathbb{N} \ . \tag{2.108}$$

The notation (2.108) means:

$$\prod_{k=0}^{k\cdot\delta<x} (x-k\cdot\delta) = x(x-\delta)(x-2\cdot\delta)\cdots(x-m\cdot\delta) \ ,$$

where m is the largest integer for which $m\cdot\delta < x$.

Examples:

$$4.37!(0.82) = \\ 4.37(4.47-0.82)(4.47-2\cdot 0.82)(4.47-3\cdot 0.82) \\ \times (4.47-4\cdot 0.82)(4.47-5\cdot 0.82) = 23.80652826961506 \ .$$

And more generally.

Let $\lambda \in \mathbb{R}$ be real number, then can consider formula:

$$x!(\delta)(\lambda) = \prod_{k=0}^{\lambda+k\cdot\delta<x} (x-k\cdot\delta) \ , \ \text{ where } \ k \in \mathbb{N} \ . \tag{2.109}$$

Examples:

1.

$$6!(1.2)(1.5) = 6(6-1.2)(6-2\cdot 1.2)(6-3\cdot 1.2) = 248.83200000000002\ ,$$

because $6-3\cdot 1.2 = 2.4 > 1.5$ and $6-4\cdot 1.2 = 1.2 < 1.5$.

2.

$$\begin{aligned} 4.37!(0.82)(-3.25) = \\ &4.37(4.47-0.82)(4.47-2\cdot 0.82)(4.47-3\cdot 0.82)\\ &\times(4.47-4\cdot 0.82)(4.47-5\cdot 0.82)(4.47-6\cdot 0.82)\\ &\times(4.47-7\cdot 0.82)(4.47-8\cdot 0.82)(4.47-9\cdot 0.82)\\ &= 118.24694616330815\ , \end{aligned}$$

3.

$$\begin{aligned} 4.37!(0.82)(-4.01) = \\ &4.37(4.47-0.82)(4.47-2\cdot 0.82)(4.47-3\cdot 0.82)\\ &\times(4.47-4\cdot 0.82)(4.47-5\cdot 0.82)(4.47-6\cdot 0.82)\\ &\times(4.47-7\cdot 0.82)(4.47-8\cdot 0.82)(4.47-9\cdot 0.82)\\ &\times(4.47-10\cdot 0.82) = -452.8858038054701\ . \end{aligned}$$

Program 2.122. the calculation of generalized factorial.

```
gf(x,δ,λ) := | return "Error." if δ < 0
             | return 1 if x=0
             | f ← x
             | k ← 1
             | while x − k·δ ≥ λ
             |    | f ← f·(x − k·δ) if x − k·δ ≠ 0
             |    | k ← k + 1
             | return f
```

This program covers all formulas given by (2.107–2.109), as you can see in the following examples:

1. $gf(7,1,0) = 5040 = 7!$,
2. $gf(2.5,1,0) = 1.875$,
3. $gf(4.37,1,0) = 17.6922054957$,

4. $gf(4.37, 0.82, 0) = 23.80652826961506$,
5. $gf(4.37, 0.82, -3.25) = 118.24694616330815$,
6. $gf(4.37, 0.82, -4.01) = -452.8858038054701$.

2.32.2 Smarandacheial

Let $n > k \geq 1$ be two integers. Then the Smarandacheial, [Smarandache and Dezert , editor], is defined as:

$$!n!_k = \prod_{i=0}^{0<|n-i\cdot k|\leq n} (n - i \cdot k) \tag{2.110}$$

For examples:

1. In the case $k = 1$:

$$\begin{aligned} !n!_1 \equiv !n! = & \prod_{i=0}^{0<|n-i|\leq n} (n-i) \\ & = n(n-1)\cdots 2\cdot 1\cdot(-1)\cdot(-2)\cdots(-n+1)(-n) = (-1)^n (n!)^2 \ . \end{aligned}$$

$$!5! = 5\cdot 4\cdot 3\cdot 2\cdot 1\cdot(-1)\cdot(-2)\cdot(-3)\cdot(-4)\cdot(-5) = -14400 = (-1)^5 120^2 \ .$$

To calculate $!n!$ can use the program *gf*, given by 2.122, as shown in the following example:

$$gf(-5, 1, -5) = -14400 \ .$$

The sequence of the first 20 numbers $!n! = gf(n, 1, -n)$ is found in following table.

Table 2.42: Smarandacheial of order 1

n	$gf(n,1,-n)$
1	–1
2	4
3	–36
4	576
5	–14400
6	518400
7	–25401600

Continued on next page

n	$gf(n,1,-n)$
8	1625702400
9	−131681894400
10	13168189440000
11	−1593350922240000
12	229442532802560000
13	−38775788043632640000
14	7600054456551997440000
15	−1710012252724199424000000
16	437763136697395052544000000
17	−126513546505547170185216000000
18	40990389067797283140009984000000
19	−14797530453474819213543604224000000
20	5919012181389927685417441689600000000

2. In case $k = 2$:

   (a) If n is odd, then

$$
\begin{aligned}
!n!_2 &= \prod_{i=0}^{0<|n-2i|\le n} (n-2i) \\
&= n(n-2)\cdots 3\cdot 1\cdot(-1)\cdot(-3)\cdots(-n+2)(-n) = (-1)^{\frac{n+1}{2}}(n!!)^2 \ .
\end{aligned}
$$

$$!5!_2 = 5(5-2)(5-4)(5-6)(5-8)(5-10) = -225 = (-1)^3 15^2 \ .$$

This result can be achieved with function *gf*, given by 2.122,

$$gf(5,2,-5) = -225 \ .$$

   (b) If n is even, then

$$
\begin{aligned}
!n!_2 &= \prod_{i=0}^{0<|n-2i|\le n} (n-2i) \\
&= n(n-2)\cdots 4\cdot 2\cdot(-2)\cdot(-4)\cdots(-n+2)(-n) = (-1)^{\frac{n}{2}}(n!!)^2 \ .
\end{aligned}
$$

$$!6!_2 = 6(6-2)(6-4)(6-8)(6-10)(6-12) = -2304 = (-1)^3 48^2,$$

This result can be achieved with function *gf*, given by 2.122,

$$gf(6,2,-6) = -2304 .$$

The sequence of the first 20 numbers $!n!_2 = gf(n,2,-n)$ is found in following table.

Table 2.43: Smarandacheial of order 2

n	$gf(n,2,-n)$
1	–1
2	–4
3	9
4	64
5	–225
6	–2304
7	11025
8	147456
9	–893025
10	–14745600
11	108056025
12	2123366400
13	–18261468225
14	–416179814400
15	4108830350625
16	106542032486400
17	–1187451971330625
18	–34519618525593600
19	428670161650355625
20	13807847410237440000

The sequence of the first 20 numbers $!n!_3 = gf(n,3,-n)$ is found in following table.

Table 2.44: Smarandacheial of order 3

n	$gf(n,3,-n)$
1	1

Continued on next page

n	$gf(n,3,-n)$
2	–2
3	–9
4	–8
5	40
6	324
7	280
8	–2240
9	–26244
10	–22400
11	246400
12	3779136
13	3203200
14	–44844800
15	–850305600
16	–717516800
17	12197785600
18	275499014400
19	231757926400
20	–4635158528000

For $n := 1..20$, one obtains:

$gf(n,4,-n)^{\mathrm{T}} \to$ 1, –4, –3, –16, –15, 144, 105, 1024, 945, –14400, –10395, –147456, –135135, 2822400, 2027025, 37748736, 34459425, –914457600, –654729075, –15099494400 ;

$gf(n,5,-n)^{\mathrm{T}} \to$ 1, 2, –6, –4, –25, –24, –42, 336, 216, 2500, 2376, 4032, –52416, –33264, –562500, –532224, –891072, 16039296, 10112256, 225000000 ;

$gf(n,6,-n)^{\mathrm{T}} \to$ 1, 2, –9, –8, –5, –36, –35, –64, 729, 640, 385, 5184, 5005, 8960, –164025, –143360, –85085, –1679616, –1616615, –2867200 ;

$gf(n,7,-n)^{\mathrm{T}} \to$ 1, 2, 3, –12, –10, –6, –49, –48, –90, –120, 1320, 1080, 624, 9604, 9360, 17280, 22440, –403920, –328320, –187200 .

We propose to proving the theorem:

Theorem 2.123. *The formula*

$$!n!_k = (-1)^{\frac{n-1-\mod(n-1,k)}{k}+1} (n \underbrace{!!\ldots!}_{k \text{ times}})^2 ,$$

for $n,k\in\mathbb{N}^*$, $n>k\ge 1$, *is true.*

Theorem 2.124. *For any integers* $k\ge 2$ *and* $n\ge k-1$ *following equality*

$$n! = k^n \prod_{i=0}^{k-1}\left(\frac{n-i}{k}\right)! \tag{2.111}$$

is true.

Proof.

Verification $k=n+1$, then

$$(n+1)^n \prod_{i=0}^{n}\left(\frac{n-i}{n+1}\right)! = (n+1)^n \prod_{i=1}^{n}\frac{n-i}{n+1} = n!\,.$$

For any $n\ge k-1$ suppose that (2.111) is true, to prove that

$$(n+1)! = k^{n+1}\prod_{i=0}^{k-1}\left(\frac{n+1-i}{k}\right)!\,.$$

Really

$$\begin{aligned}(n+1)! = (n+1)n! = (n+1)k^n \prod_{i=0}^{k-1}\left(\frac{n-i}{k}\right)!\\
&= k^{n+1}\frac{n+1}{k}\left(\frac{n-k+1}{k}\right)!\prod_{i=0}^{k-2}\left(\frac{n-i}{k}\right)!\\
&= k^{n+1}\frac{n+1}{k}\left(\frac{n+1}{k}-1\right)!\prod_{i=0}^{k-2}\left(\frac{n-i}{k}\right)!\\
&= k^{n+1}\left(\frac{n+1}{k}\right)!\prod_{i=0}^{k-2}\left(\frac{n-i}{k}\right)! = k^{n+1}\prod_{i=0}^{k-1}\left(\frac{n+1-i}{k}\right)!\,.\end{aligned}$$

□

2.33 Analogues of the Smarandache Function

Let $a:\mathbb{N}^*\to\mathbb{N}^*$ be a function, where $a(n)$ is the smallest number m such that $n\le m!$ [Yuan and Wenpeng, 2005], [Sloane, 2014, A092118].

Program 2.125. for function a.

$$a(n) := \left|\begin{array}{l} for\ \ m\in 1..1000\\ \quad return\ \ m\ \ if\ \ m!\ge n\\ return\ \ \text{"}Error.\text{"}\end{array}\right.$$

For $n := 1..25$ one obtains $a(n) =$ 1, 1, 2, 3, 3, 3, 3, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 5 and $a(10^{10}) \to 14$, $a(10^{20}) \to 22$, $a(10^{30}) \to 29$, $a(10^{40}) \to 35$, $a(10^{50}) \to 42$, $a(10^{60}) \to 48$, $a(10^{70}) \to 54$, $a(10^{80}) \to 59$, $a(10^{90}) \to 65$, $a(10^{100}) \to 70$.

2.34 Power Function

2.34.1 Power Function of Second Order

The function $SP2 : \mathbb{N}^* \to \mathbb{N}^*$, where $SP2(n)$ is the smallest number m such that m^m is divisible by n.

Program 2.126. for the function *SP2*.

$$SP2(n) := \left| \begin{array}{l} for\ \ m \in 1..n \\ return\ \ m\ \ if\ \ \ \mathrm{mod}\,(m^m, n) = 0 \end{array} \right.$$

For $n := 1..10^2$, the command $sp2_n := SP2(n)$, generate the sequence $sp2^{\mathrm{T}} \to$(1 2 3 2 5 6 7 4 3 10 11 6 13 14 15 4 17 6 19 10 21 22 23 6 5 26 3 14 29 30 31 4 33 34 35 6 37 38 39 10 41 42 43 22 15 46 47 6 7 10 51 26 53 6 55 14 57 58 59 30 61 62 21 4 65 66 67 34 69 70 71 6 73 74 15 38 77 78 79 10 6 82 83 42 85 86 87 22 89 30 91 46 93 94 95 6 97 14 33 10).

Remark 2.127. relating to function *SP2*, [Smarandache, 1998, Xu, 2006, Zhou, 2006].

1. If $p \in \mathbb{P}_{\geq 2}$, then $SP2(p) = p$;
2. If r is square free, then $SP2(r) = r$;
3. If $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_s^{\alpha_s}$ and $\alpha_k \leq p_k$, for $k = 1, 2, \ldots s$, then $SP2(n) = n$;
4. If $n = p^\alpha$, where $p \in \mathbb{P}_{\geq 2}$, then:

$$SP2(n) = \begin{cases} p & \text{if } 1 \leq \alpha \leq p\,, \\ p^2 & \text{if } p+1 \leq \alpha \leq 2 \cdot p^2\,, \\ p^3 & \text{if } 2p^2+1 \leq \alpha \leq 3 \cdot p^3\,, \\ \vdots & \vdots \\ p^s & \text{if } (s-1)p^{s-1}+1 \leq \alpha \leq s \cdot p^s\,. \end{cases}$$

2.34.2 Power Function of Third Order

The function $SP3 : \mathbb{N}^* \to \mathbb{N}^*$, where $SP3(n)$ is the smallest number m such that m^{m^m} is divisible by n.

Program 2.128. for the function *SP3*.

$$SP3(n) := \left| \begin{array}{l} for\ \ m \in 1..n \\ return\ \ m\ \ if\ \ \bmod\left(m^{m^m}, n\right) = 0 \end{array} \right.$$

For $n := 1..10^2$, the command $sp3_n := SP3(n)$, generate the sequence $sp3^{\mathrm{T}} \to$(1 2 3 2 5 6 7 2 3 10 11 6 13 14 15 2 17 6 19 10 21 22 23 6 5 26 3 14 29 30 31 4 33 34 35 6 37 38 39 10 41 42 43 22 15 46 47 6 7 10 51 26 53 6 55 14 57 58 59 30 61 62 21 4 65 66 67 34 69 70 71 6 73 74 15 38 77 78 79 10 3 82 83 42 85 86 87 22 89 30 91 46 93 94 95 6 97 14 33 10).

Chapter 3

Sequences of Numbers Involved in Unsolved Problems

Here it is a long list of sequences, functions, unsolved problems, conjectures, theorems, relationships, operations, etc. Some of them are inter–connected [Knuth, 2005], [Sloane, 2014], [Smarandache, 1993b].

3.1 Consecutive Sequence

How many primes are there among these numbers? In a general form, the consecutive sequence is considered in an arbitrary numeration base b? [Smarandache, 2014, 1979]

Table 3.1: Consecutive sequence

#	$n_{(10)}$
1	1
2	12
3	123
4	1234
5	12345
6	123456
7	1234567
8	12345678
9	123456789
10	12345678910
11	1234567891011

Continued on next page

#	$n_{(10)}$
12	123456789101112
13	12345678910111213
14	1234567891011121314
15	123456789101112131415
16	12345678910111213141516
17	1234567891011121314151617
18	123456789101112131415161718
19	12345678910111213141516171819
20	1234567891011121314151617181920
21	123456789101112131415161718192021
22	12345678910111213141516171819202122
23	1234567891011121314151617181920212223
24	123456789101112131415161718192021222324
25	12345678910111213141516171819202122232425
26	1234567891011121314151617181920212223242526

Table 3.2: Factored consecutive sequence

#	factors
1	1
2	$2^2 \cdot 3$
3	$3 \cdot 41$
4	$2 \cdot 617$
5	$3 \cdot 5 \cdot 823$
6	$2^6 \cdot 3 \cdot 643$
7	$127 \cdot 9721$
8	$2 \cdot 3^2 \cdot 47 \cdot 14593$
9	$3^2 \cdot 3607 \cdot 3803$
10	$2 \cdot 5 \cdot 1234567891$
11	$3 \cdot 7 \cdot 13 \cdot 67 \cdot 107 \cdot 630803$
12	$2^3 \cdot 3 \cdot 2437 \cdot 2110805449$
13	$113 \cdot 125693 \cdot 869211457$
14	$2 \cdot 3 \cdot 205761315168520219$
15	$3 \cdot 5 \cdot 8230452606740808761$
16	$2^2 \cdot 2507191691 \cdot 1231026625769$
17	$3^2 \cdot 47 \cdot 4993 \cdot 584538396786764503$
18	$2 \cdot 3^2 \cdot 97 \cdot 88241 \cdot 801309546900123763$

Continued on next page

#	factors
19	$13 \cdot 43 \cdot 79 \cdot 281 \cdot 1193 \cdot 833929457045867563$
20	$2^5 \cdot 3 \cdot 5 \cdot 323339 \cdot 3347983 \cdot 2375923237887317$
21	$3 \cdot 17 \cdot 37 \cdot 43 \cdot 103 \cdot 131 \cdot 140453 \cdot 802851238177109689$
22	$2 \cdot 7 \cdot 1427 \cdot 3169 \cdot 85829 \cdot 2271991367799686681549$
23	$3 \cdot 41 \cdot 769 \cdot 13052194181136110820214375991629$
24	$2^2 \cdot 3 \cdot 7 \cdot 978770977394515241 \cdot 1501601205715706321$
25	$5^2 \cdot 15461 \cdot 31309647077 \cdot 1020138683879280489689401$
26	$2 \cdot 3^4 \cdot 21347 \cdot 2345807 \cdot 982658598563 \cdot 154870313069150249$

In base 10, with the "digits" $\in \{1, 2, \ldots, 26\}$ not are primes.

Table 3.3: Binary consecutive sequence in base 2

#	$n_{(2)}$
1	1
2	110
3	11011
4	11011100
5	11011100101
6	11011100101110
7	11011100101110111
8	110111001011101111000
9	1101110010111011110001001
10	11011100101110111100010011010
11	110111001011101111000100110101011
12	1101110010111011110001001101010111100
13	11011100101110111100010011010101111001101
14	110111001011101111000100110101011110011011110
15	1101110010111011110001001101010111100110111101111

Table 3.4: Binary consecutive sequence in base 10

#	$n_{(10)}$	factors
1	1	1

Continued on next page

#	$n_{(10)}$	factors
2	6	$2 \cdot 3$
3	27	3^3
4	220	$2^2 \cdot 5 \cdot 11$
5	1765	$5 \cdot 353$
6	14126	$2 \cdot 7 \cdot 1009$
7	113015	$5 \cdot 7 \cdot 3229$
8	1808248	$2^3 \cdot 13 \cdot 17387$
9	28931977	$17 \cdot 1701881$
10	462911642	$2 \cdot 167 \cdot 1385963$
11	7406586283	$29 \cdot 53 \cdot 179 \cdot 26921$
12	118505380540	$2^2 \cdot 5 \cdot 5925269027$
13	1896086088653	$109 \cdot 509 \cdot 2971 \cdot 11503$
14	30337377418462	$2 \cdot 15168688709231$
15	485398038695407	$\boxed{485398038695407}$

The numbers given in the box are prime numbers. In base 2, with the "digits" $\in \{1,2,\ldots,15\}$ the number $\boxed{485398038695407}$ is a prime number.

Table 3.5: Ternary consecutive sequence in base 3

#	$n_{(3)}$
1	1
2	12
3	1210
4	121011
5	12101120
6	1210112021
7	121011202122
8	121011202122100
9	121011202122100101
10	121011202122100101110
11	121011202122100101110111
12	121011202122100101110111200
13	121011202122100101110111200201
14	121011202122100101110111200201210
15	121011202122100101110111200201210211
16	121011202122100101110111200201210211220

Table 3.6: Ternary consecutive sequence in base 10

#	$n_{(10)}$	factors
1	1	1
2	5	5
3	48	$2^4 \cdot 3$
4	436	$2^2 \cdot 109$
5	3929	$\boxed{3929}$
6	35367	$3 \cdot 11789$
7	318310	$2 \cdot 5 \cdot 139 \cdot 229$
8	2864798	$2 \cdot 97 \cdot 14767$
9	77349555	$3^2 \cdot 5 \cdot 1718879$
10	2088437995	$5 \cdot 7 \cdot 59669657$
11	56387825876	$2^2 \cdot 14096956469$
12	1522471298664	$2^3 \cdot 3 \cdot 63436304111$
13	41106725063941	$6551 \cdot 11471 \cdot 547021$
14	1109881576726421	$41 \cdot 27070282359181$
15	29966802571613382	$2 \cdot 3 \cdot 17 \cdot 2935459 \cdot 100083899$

In base 3, with the "digits" $\in \{1, 2, \ldots, 15\}$ number $\boxed{3929}$ is prime number.

Table 3.7: Octal consecutive sequence

$n_{(8)}$	$n_{(10)}$	factors
1	1	1
12	10	$2 \cdot 5$
123	83	$\boxed{83}$
1234	668	$2^2 \cdot 167$
12345	5349	$3 \cdot 1783$
123456	42798	$2 \cdot 3 \cdot 7 \cdot 1019$
1234567	342391	$7 \cdot 41 \cdot 1193$
123456710	21913032	$2^3 \cdot 3 \cdot 31 \cdot 29453$
12345671011	1402434057	$3 \cdot 17^2 \cdot 157 \cdot 10303$
1234567101112	89755779658	$2 \cdot 44877889829$
123456710111213	5744369898123	$3 \cdot 83 \cdot 23069758627$

Continued on next page

$n_{(8)}$	$n_{(10)}$	factors
12345671011121314	367639673479884	$2^2 \cdot 3^2 \cdot 13 \cdot 29 \cdot 53 \cdot 511096199$
1234567101112131415	23528939102712588	$2^2 \cdot 3 \cdot 461 \cdot 4253242787909$

In octal, with the "digits" $\in \{1, 2, \ldots, 15\}$ only number $\boxed{83}$ is prime number.

Table 3.8: Hexadecimal consecutive sequence

$n_{(16)}$	$n_{(10)}$	factors
1	1	1
12	18	$2 \cdot 3^2$
123	291	$3 \cdot 97$
1234	4660	$2^2 \cdot 5 \cdot 233$
12345	74565	$3^2 \cdot 5 \cdot 1657$
123456	1193046	$2 \cdot 3 \cdot 198841$
1234567	19088743	$2621 \cdot 7283$
12345678	305419896	$2^3 \cdot 3^5 \cdot 157109$
123456789	4886718345	$3^2 \cdot 5 \cdot 23 \cdot 4721467$
123456789a	78187493530	$2 \cdot 5 \cdot 7818749353$
123456789ab	1250999896491	$3^2 \cdot 12697 \cdot 10947467$
123456789abc	20015998343868	$2^2 \cdot 3 \cdot 1242757 \cdot 1342177$
123456789abcd	320255973501901	$\boxed{320255973501901}$
123456789abcde	5124095576030430	$2 \cdot 3^2 \cdot 5 \cdot 215521 \cdot 264170987$

In hexadecimal, with the "digits" $\in \{1, 2, \ldots, 15\}$ number $\boxed{320255973501901}$ is prime.

3.2 Circular Sequence

Table 3.9: Circular sequence

$n_{(10)}$	factors	$n_{(10)}$	factors
12	$2^2 \cdot 3$	13245	$3 \cdot 5 \cdot 883$
21	$3 \cdot 7$	13254	$2 \cdot 3 \cdot 47^2$

Continued on next page

$n_{(10)}$	factors	$n_{(10)}$	factors
123	$3\cdot41$	13425	$3\cdot5^2\cdot179$
132	$2^2\cdot3\cdot11$	13452	$2^2\cdot3\cdot19\cdot59$
213	$3\cdot71$	13524	$2^2\cdot3\cdot7^2\cdot23$
231	$3\cdot7\cdot11$	13542	$2\cdot3\cdot37\cdot61$
312	$2^3\cdot3\cdot13$	14235	$3\cdot5\cdot13\cdot73$
321	$3\cdot107$	14253	$3\cdot4751$
1234	$2\cdot617$	14325	$3\cdot5^2\cdot191$
1243	$11\cdot113$	14352	$2^4\cdot3\cdot13\cdot23$
1324	$2^2\cdot331$	14523	$3\cdot47\cdot103$
1342	$2\cdot11\cdot61$	14532	$2^2\cdot3\cdot7\cdot173$
1423	$\boxed{1423}$	15234	$2\cdot3\cdot2539$
1432	$2^3\cdot179$	15243	$3\cdot5081$
2134	$2\cdot11\cdot97$	15324	$2^2\cdot3\cdot1277$
2143	$\boxed{2143}$	15342	$2\cdot3\cdot2557$
2314	$2\cdot13\cdot89$	15423	$3\cdot53\cdot97$
2341	$\boxed{2341}$	15432	$2^3\cdot3\cdot643$
2413	$19\cdot127$	21345	$3\cdot5\cdot1423$
2431	$11\cdot13\cdot17$	21354	$2\cdot3\cdot3559$
3124	$2^2\cdot11\cdot71$	21435	$3\cdot5\cdot1429$
3142	$2\cdot1571$	21453	$3\cdot7151$
3214	$2\cdot1607$	21534	$2\cdot3\cdot37\cdot97$
3241	$7\cdot463$	21543	$3\cdot43\cdot167$
3412	$2^2\cdot853$	23145	$3\cdot5\cdot1543$
3421	$11\cdot311$	23154	$2\cdot3\cdot17\cdot227$
4123	$7\cdot19\cdot31$	23415	$3\cdot5\cdot7\cdot223$
4132	$2^2\cdot1033$	23451	$3\cdot7817$
4213	$11\cdot383$	23514	$2\cdot3\cdot3919$
4231	$\boxed{4231}$	23541	$3\cdot7\cdot19\cdot59$
4312	$2^3\cdot7^2\cdot11$	24135	$3\cdot5\cdot1609$
4321	$29\cdot149$	24153	$3\cdot83\cdot97$
12345	$3\cdot5\cdot823$	24315	$3\cdot5\cdot1621$
12354	$2\cdot3\cdot29\cdot71$	24351	$3\cdot8117$
12435	$3\cdot5\cdot829$	24513	$3\cdot8171$
12453	$3\cdot7\cdot593$	24531	$3\cdot13\cdot17\cdot37$
12534	$2\cdot3\cdot2089$	25134	$2\cdot3\cdot59\cdot71$
12543	$3\cdot37\cdot113$	25143	$3\cdot17^2\cdot29$
25314	$2\cdot3\cdot4219$	42513	$3\cdot37\cdot383$
25341	$3\cdot8447$	42531	$3\cdot14177$
25413	$3\cdot43\cdot197$	43125	$3\cdot5^4\cdot23$

Continued on next page

$n_{(10)}$	factors	$n_{(10)}$	factors
25431	$3 \cdot 7^2 \cdot 173$	43152	$2^4 \cdot 3 \cdot 29 \cdot 31$
31245	$3 \cdot 5 \cdot 2083$	43215	$3 \cdot 5 \cdot 43 \cdot 67$
31254	$2 \cdot 3 \cdot 5209$	43251	$3 \cdot 13 \cdot 1109$
31425	$3 \cdot 5^2 \cdot 419$	43512	$2^3 \cdot 3 \cdot 7^2 \cdot 37$
31452	$2^2 \cdot 3 \cdot 2621$	43521	$3 \cdot 89 \cdot 163$
31524	$2^2 \cdot 3 \cdot 37 \cdot 71$	45123	$3 \cdot 13^2 \cdot 89$
31542	$2 \cdot 3 \cdot 7 \cdot 751$	45132	$2^2 \cdot 3 \cdot 3761$
32145	$3 \cdot 5 \cdot 2143$	45213	$3 \cdot 7 \cdot 2153$
32154	$2 \cdot 3 \cdot 23 \cdot 233$	45231	$3 \cdot 15077$
32415	$3 \cdot 5 \cdot 2161$	45312	$2^8 \cdot 3 \cdot 59$
32451	$3 \cdot 29 \cdot 373$	45321	$3 \cdot 15107$
32514	$2 \cdot 3 \cdot 5419$	51234	$2 \cdot 3 \cdot 8539$
32541	$3 \cdot 10847$	51243	$3 \cdot 19 \cdot 29 \cdot 31$
34125	$3 \cdot 5^3 \cdot 7 \cdot 13$	51324	$2^2 \cdot 3 \cdot 7 \cdot 13 \cdot 47$
34152	$2^3 \cdot 3 \cdot 1423$	51342	$2 \cdot 3 \cdot 43 \cdot 199$
34215	$3 \cdot 5 \cdot 2281$	51423	$3 \cdot 61 \cdot 281$
34251	$3 \cdot 7^2 \cdot 233$	51432	$2^3 \cdot 3 \cdot 2143$
34512	$2^4 \cdot 3 \cdot 719$	52134	$2 \cdot 3 \cdot 8689$
34521	$3 \cdot 37 \cdot 311$	52143	$3 \cdot 7 \cdot 13 \cdot 191$
35124	$2^2 \cdot 3 \cdot 2927$	52314	$2 \cdot 3 \cdot 8719$
35142	$2 \cdot 3 \cdot 5857$	52341	$3 \cdot 73 \cdot 239$
35214	$2 \cdot 3 \cdot 5869$	52413	$3 \cdot 17471$
35241	$3 \cdot 17 \cdot 691$	52431	$3 \cdot 17477$
35412	$2^2 \cdot 3 \cdot 13 \cdot 227$	53124	$2^2 \cdot 3 \cdot 19 \cdot 233$
35421	$3 \cdot 11807$	53142	$2 \cdot 3 \cdot 17 \cdot 521$
41235	$3 \cdot 5 \cdot 2749$	53214	$2 \cdot 3 \cdot 7^2 \cdot 181$
41253	$3 \cdot 13751$	53241	$3 \cdot 17747$
41325	$3 \cdot 5^2 \cdot 19 \cdot 29$	53412	$2^2 \cdot 3 \cdot 4451$
41352	$2^3 \cdot 3 \cdot 1723$	53421	$3 \cdot 17807$
41523	$3 \cdot 13841$	54123	$3 \cdot 18041$
41532	$2^2 \cdot 3 \cdot 3461$	54132	$2^2 \cdot 3 \cdot 13 \cdot 347$
42135	$3 \cdot 5 \cdot 53^2$	54213	$3 \cdot 17 \cdot 1063$
42153	$3 \cdot 14051$	54231	$3 \cdot 18077$
42315	$3 \cdot 5 \cdot 7 \cdot 13 \cdot 31$	54312	$2^3 \cdot 3 \cdot 31 \cdot 73$
42351	$3 \cdot 19 \cdot 743$	54321	$3 \cdot 19 \cdot 953$

The numbers $\boxed{1423}$, $\boxed{2143}$, $\boxed{2341}$ and $\boxed{4231}$ are the only primes for circular sequences 12, 21, 123, ..., 321, 1234, ..., 54321 .

3.3 Symmetric Sequence

The sequence of symmetrical numbers was considered in the works [Smarandache, 1979, 2014]

Table 3.10: Symmetric sequence

$n_{(10)}$	factors
1	1
11	$\boxed{11}$
121	11^2
1221	$3 \cdot 11 \cdot 37$
12321	$3^2 \cdot 37^2$
123321	$3 \cdot 11 \cdot 37 \cdot 101$
1234321	$11^2 \cdot 101^2$
12344321	$11 \cdot 41 \cdot 101 \cdot 271$
123454321	$41^2 \cdot 271^2$
1234554321	$3 \cdot 7 \cdot 11 \cdot 13 \cdot 37 \cdot 41 \cdot 271$
12345654321	$3^2 \cdot 7^2 \cdot 11^2 \cdot 13^2 \cdot 37^2$
123456654321	$3 \cdot 7 \cdot 11 \cdot 13 \cdot 37 \cdot 239 \cdot 4649$
1234567654321	$239^2 \cdot 4649^2$
12345677654321	$11 \cdot 73 \cdot 101 \cdot 137 \cdot 239 \cdot 4649$
123456787654321	$11^2 \cdot 73^2 \cdot 101^2 \cdot 137^2$
1234567887654321	$3^2 \cdot 11 \cdot 37 \cdot 73 \cdot 101 \cdot 137 \cdot 333667$
12345678987654321	$3^4 \cdot 37^2 \cdot 333667^2$
123456789987654321	$3^2 \cdot 11 \cdot 37 \cdot 41 \cdot 271 \cdot 9091 \cdot 333667$
12345678910987654321	$\boxed{12345678910987654321}$
1234567891010987654321	$\boxed{1234567891010987654321}$
123456789101110987654321	$7 \cdot 17636684157301569664903$
12345678910111110987654321	$3 \cdot 43 \cdot 97 \cdot 548687 \cdot 1798162193492191$
1234567891011121110987654321	$3^2 \cdot 7^2 \cdot 2799473675762179389994681$

3.4 Deconstructive Sequence

Deconstructive sequence with the decimal digits $\{1, 2, \ldots, 9\}$, [Smarandache, 1993a, 2014].

Table 3.11: Deconstructive sequence with $\{1, 2, \ldots, 9\}$

$n_{(10)}$	factors
1	1
23	$\boxed{23}$
456	$2^3 \cdot 3 \cdot 19$
7891	$13 \cdot 607$
23456	$2^5 \cdot 733$
789123	$3 \cdot 17 \cdot 15473$
4567891	$\boxed{4567891}$
23456789	$\boxed{23456789}$
123456789	$3^2 \cdot 3607 \cdot 3803$
1234567891	$\boxed{1234567891}$
23456789123	$59 \cdot 397572697$
456789123456	$2^7 \cdot 3 \cdot 23 \cdot 467 \cdot 110749$
7891234567891	$37 \cdot 353 \cdot 604183031$
23456789123456	$2^7 \cdot 13 \cdot 23 \cdot 47 \cdot 13040359$
789123456789123	$3 \cdot 19 \cdot 13844271171739$
4567891234567891	$739 \cdot 1231 \cdot 4621 \cdot 1086619$
23456789123456789	$\boxed{23456789123456788}$
123456789123456789	$3^2 \cdot 7 \cdot 11 \cdot 13 \cdot 19 \cdot 3607$
	$\times 3803 \cdot 52579$
1234567891234567891	$31 \cdot 241 \cdot 1019 \cdot 162166841159$

Deconstructive sequence with the decimal digits $\{1, 2, \ldots, 9, 0\}$.

Table 3.12: Deconstructive sequence with $\{1, 2, \ldots, 9, 0\}$

$n_{(10)}$	factors
1	1
23	$\boxed{23}$
456	$2^3 \cdot 3 \cdot 19$
7890	$2 \cdot 3 \cdot 5 \cdot 263$
12345	$3 \cdot 5 \cdot 823$
678901	$\boxed{678901}$
2345678	$2 \cdot 23 \cdot 50993$
90123456	$2^6 \cdot 3 \cdot 367 \cdot 1279$

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$n_{(10)}$	factors
789012345	$3 \cdot 5 \cdot 11 \cdot 131 \cdot 173 \cdot 211$
6789012345	$3^2 \cdot 5 \cdot 150866941$
67890123456	$2^6 \cdot 3 \cdot 353594393$
789012345678	$2 \cdot 3 \cdot 19 \cdot 9133 \cdot 757819$
9012345678901	$\boxed{9012345678901}$
23456789012345	$5 \cdot 13 \cdot 19 \cdot 89 \cdot 213408443$
678901234567890	$2 \cdot 3 \cdot 5 \cdot 1901 \cdot 11904282563$
1234567890123456	$2^6 \cdot 3 \cdot 7^2 \cdot 301319 \cdot 435503$

3.5 Concatenated Sequences

Sequences obtained from concatenating the sequences of numbers: primes, Fibonacci, Mersenne, etc.

Program 3.1. for concatenation the terms of sequence.

$$ConS(s,L) := \left| \begin{array}{l} cs_1 \leftarrow s_1 \\ for\ k \in 2..L \\ \quad cs_k \leftarrow conc(cs_{k-1}, s_k) \\ return\ \ cs \end{array} \right.$$

Program 3.2. for back concatenation the terms of sequence.

$$BConS(s,L) := \left| \begin{array}{l} cs_1 \leftarrow s_1 \\ for\ k \in 2..L \\ \quad cs_k \leftarrow conc(s_k, cs_{k-1}) \\ return\ \ cs \end{array} \right.$$

It was obtained by the programs *ConS*, 3.1 and *BConS*, 3.2 using the routine *conc*, 7.1.

3.5.1 Concatenated Prime Sequence

Using the program 3.1 one can generate a Concatenated Prime Sequence (called Smarandache–Wellin numbers)

$$L := 20 \quad p := submatrix(prime, 1, L, 1, 1) \quad cp := ConS(p, L)$$

then $cp \rightarrow$ provides the vector:

Table 3.13: Concatenated Prime Sequence

k	cp_k
1	2
2	23
3	235
4	2357
5	235711
6	23571113
7	2357111317
8	235711131719
9	23571113171923
10	2357111317192329
11	235711131719232931
12	23571113171923293137
13	2357111317192329313741
14	235711131719232931374143
15	23571113171923293137414347
16	2357111317192329313741434753
17	235711131719232931374143475359
18	23571113171923293137414347535961
19	2357111317192329313741434753596167
20	235711131719232931374143475359616771

Factorization of the vector cp is obtained with the command: cp *factor*→

Table 3.14: Factorization Concatenated Prime Sequence

k	cp_k
1	$\boxed{2}$
2	$\boxed{23}$
3	$5 \cdot 47$
4	$\boxed{2357}$
5	$7 \cdot 151 \cdot 223$
6	$23 \cdot 29 \cdot 35339$
7	$11 \cdot 214282847$
8	$7 \cdot 4363 \cdot 7717859$

Continued on next page

k	cp_k
9	$61 \cdot 478943 \cdot 806801$
10	$3 \cdot 4243 \cdot 185176472401$
11	$17 \cdot 283 \cdot 1787 \cdot 76753 \cdot 357211$
12	$7 \cdot 67^2 \cdot 151 \cdot 4967701595369$
13	$25391 \cdot 889501 \cdot 104364752351$
14	$6899 \cdot 164963 \cdot 7515281 \cdot 27558919$
15	$1597 \cdot 2801 \cdot 5269410931806332951$
16	$3 \cdot 2311 \cdot 1237278209 \cdot 274784055330749$
17	$17 \cdot 906133 \cdot 12846401 \cdot 1191126125288819$
18	$3 \cdot 13 \cdot 3390511326677 \cdot 178258515898000387$
19	$1019 \cdot 2313161253378144566969023310693$
20	$3^3 \cdot 873004191552714560644975834665 2473$

3.5.2 Back Concatenated Prime Sequence

Using the program 3.2 one can generate a Back Concatenated Prime Sequence (on short BCPS) $L := 20$ $p := submatrix(prime, 2, L+1, 1, 1)^{\mathrm{T}} = (3\ 5\ 7\ 11\ 13\ 17\ 19\ 23\ 29\ 31\ 37\ 41\ 43\ 47\ 53\ 59\ 61\ 67\ 71\ 73\ 79)$, $bcp := BConS(p, L)$, then $bcp \rightarrow$ provides the vector:

Table 3.15: Back Concatenated Prime Sequence

k	bcp_k
1	3
2	53
3	753
4	11753
5	1311753
6	171311753
7	19171311753
8	2319171311753
9	292319171311753
10	31292319171311753
11	3731292319171311753
12	413731292319171311753
13	43413731292319171311753

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k	bcp_k
14	4743413731292319171311753
15	534743413731292319171311753
16	59534743413731292319171311753
17	6159534743413731292319171311753
18	676159534743413731292319171311753
19	71676159534743413731292319171311753
20	7371676159534743413731292319171311753

Table 3.16: Factorization BCPS

k	bcp_k
1	$\boxed{3}$
2	$\boxed{53}$
3	$3 \cdot 251$
4	$7 \cdot 23 \cdot 73$
5	$3 \cdot 331 \cdot 1321$
6	$\boxed{171311753}$
7	$3 \cdot 11^2 \cdot 52813531$
8	$19 \cdot 122061647987$
9	$75041 \cdot 3895459433$
10	$463 \cdot 44683 \cdot 1512566357$
11	$3 \cdot 15913 \cdot 1110103 \cdot 70408109$
12	$17 \cdot 347 \cdot 1613 \cdot 1709 \cdot 80449 \cdot 316259$
13	$3^2 \cdot 41 \cdot 557 \cdot 1260419 \cdot 167583251039$
14	$17^2 \cdot 37 \cdot 127309607 \cdot 3484418108803$
15	$67 \cdot 241249 \cdot 33083017882204960291$
16	$3 \cdot 7 \cdot 11 \cdot 13 \cdot 281 \cdot 15289778873 \cdot 4614319153627$
17	$1786103719753 \cdot 3448587377817864001$
18	$17 \cdot 83407 \cdot 336314747 \cdot 1417920375788952821$
19	$19989277303 \cdot 3585730411773627378513151$
20	$1613 \cdot 24574819 \cdot 75164149139 \cdot 2474177239668341$

3.5.3 Concatenated Fibonacci Sequence

With the commands: $L := 20$ $f_1 := 1$ $f_2 := 1$ $k := 3..L$ $f_k := f_{k-1} + f_{k-2}$ $cF :=$ $ConS(f, l)$, resulting the vector:

Table 3.17: Concatenated Fibonacci Sequence

k	cF_k
1	1
2	$\boxed{11}$
3	112
4	$\boxed{1123}$
5	11235
6	112358
7	11235813
8	1123581321
9	112358132134
10	11235813213455
11	1123581321345589
12	1123581321345589144
13	1123581321345589144233
14	1123581321345589144233377
15	1123581321345589144233377610
16	1123581321345589144233377610987
17	11235813213455891442333776109871597
18	112358132134558914423337761098715972584
19	1123581321345589144233377610987159725844181
20	11235813213455891442333776109871597258441816765

where numbers in $\boxed{11}$, $\boxed{1123}$ are primes, [Smarandache, 1975, Marimutha, 1997, Smarandache, 1997].

3.5.4 Back Concatenated Fibonacci Sequence

With the commands: $L := 20$ $f_1 := 1$ $f_2 := 1$ $k := 3..L$ $f_k := f_{k-1} + f_{k-2}$ $bcF :=$ $BConS(f, l)$, resulting the vector:

Table 3.18: Back Concatenated Fibonacci Sequence

k	bcF_k
1	1
2	$\boxed{11}$
3	$\boxed{211}$
4	3211
5	53211
6	$\boxed{853211}$
7	13853211
8	2113853211
9	342113853211
10	55342113853211
11	8955342113853211
12	1448955342113853211
13	2331448955342113853211
14	3772331448955342113853211
15	6103772331448955342113853211
16	9876103772331448955342113853211
17	15979876103772331448955342113853211
18	258415979876103772331448955342113853211
19	4181258415979876103772331448955342113853211
20	67654181258415979876103772331448955342113853211

3.5.5 Concatenated Tetranacci Sequence

With the commands: $L := 20$ $t_1 := 1$ $t_2 := 1$ $t_3 := 2$ $k := 4..L$ $t_k := t_{k-1} + t_{k-2} + t_{k-3}$ $ct := ConS(t, L)$ $bct := BConS(t, L)$ resulting the vectors ct and bct.

Table 3.19: Concatenated Tetranacci Sequence

k	ct_k
1	1
2	$\boxed{11}$
3	112
4	1124
5	11247

Continued on next page

k	ct_k
6	1124713
7	112471324
8	11247132444
9	1124713244481
10	1124713244481149
11	1124713244481149274
12	1124713244481149274504
13	1124713244481149274504927
14	11247132444811492745049271705
15	112471324448114927450492717053136
16	1124713244481149274504927170531365768
17	112471324448114927450492717053136576810609
18	11247132444811492745049271705313657681060919513
19	1124713244481149274504927170531365768106091951335890
20	112471324448114927450492717053136576810609195133589066012

Table 3.20: Back Concatenated Tetranacci Sequence

k	bct_k
1	1
2	$\boxed{11}$
3	$\boxed{211}$
4	$\boxed{4211}$
5	74211
6	$\boxed{1374211}$
7	241374211
8	44241374211
9	8144241374211
10	1498144241374211
11	2741498144241374211
12	5042741498144241374211
13	9275042741498144241374211
14	17059275042741498144241374211
15	313617059275042741498144241374211
16	5768313617059275042741498144241374211
17	106095768313617059275042741498144241374211

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k	bct_k
18	195131060957683136170592750427414981442413742 11
19	3589019513106095768313617059275042741498144241374211
20	66012358901951310609576831361705927504274149814424137421 1

3.5.6 Concatenated Mersenne Sequence

With commands: $L := 17$ $k := 1..L$ $\ell M_k := 2^k - 1$ $rM_k := 2^k + 1$ $c\ell M := ConS(M\ell, L)$ $crM := ConS(rM, L)$ $bcM\ell := BConS(\ell M, L)$ $bcrM := BConS(Mr, L)$ resulting files:

Table 3.21: Concatenated Left Mersenne Sequence

k	$c\ell M_k$
1	1
2	$\boxed{13}$
3	$\boxed{137}$
4	13715
5	1371531
6	137153163
7	137153163127
8	137153163127255
9	$\boxed{137153163127255511}$
10	1371531631272555111023
11	13715316312725551110232047
12	137153163127255511102320474095
13	1371531631272555111023204740958191
14	137153163127255511102320474095819116383
15	13715316312725551110232047409581911638332767
16	1371531631272555111023204740958191163833276765535
17	137153163127255511102320474095819116383327676553513107 1

Table 3.22: Back Concatenated Left Mersenne Sequence

k	$bc\ell M_k$
1	1
2	$\boxed{31}$
3	731
4	$\boxed{15731}$
5	3115731
6	633115731
7	127633115731
8	255127633115731
9	511255127633115731
10	1023511255127633115731
11	20471023511255127633115731
12	409520471023511255127633115731
13	8191409520471023511255127633115731
14	163838191409520471023511255127633115731
15	32767163838191409520471023511255127633115731
16	6553532767163838191409520471023511255127633115731
17	1310716553532767163838191409520471023511255127633115731

Table 3.23: Concatenated Right Mersenne Sequence

k	crM_k
1	$\boxed{3}$
2	35
3	$\boxed{359}$
4	35917
5	$\boxed{3591733}$
6	359173365
7	359173365129
8	359173365129257
9	$\boxed{359173365129257513}$
10	3591733651292575131025
11	35917336512925751310252049
12	359173365129257513102520494097
13	3591733651292575131025204940978193

Continued on next page

k	crM_k
14	359173365129257513102520494097819316385
15	35917336512925751310252049409781931638532769
16	3591733651292575131025204940978193163853276965537
17	3591733651292575131025204940978193163853276965537131073

Table 3.24: Back Concatenated Right Mersenne Sequence

k	$bcrM_k$
1	$\boxed{3}$
2	$\boxed{53}$
3	$\boxed{953}$
4	17953
5	3317953
6	653317953
7	129653317953
8	257129653317953
9	513257129653317953
10	1025513257129653317953
11	20491025513257129653317953
12	409720491025513257129653317953
13	8193409720491025513257129653317953
14	163858193409720491025513257129653317953
15	32769163858193409720491025513257129653317953
16	6553732769163858193409720491025513257129653317953
17	1310736553732769163858193409720491025513257129653317953

3.5.7 Concatenated $6k-5$ Sequence

With commands: $L := 25$ $k := 1..L$ $six_k := 6k-5$ $six^{\mathrm{T}} = (1\ 7\ 13\ 19\ 25\ 31\ 37\ 43\ 49\ 55\ 61\ 67\ 73\ 79\ 85\ 91\ 97\ 103\ 109\ 115\ 121\ 127\ 133\ 139\ 145)$ result files $c6 := ConS(six, L)$ and $bc6 := BConS(six, L)$.

Table 3.25: Concatenated $c6$ Sequence

k	$c6_k$
1	1
2	$\boxed{17}$
3	1713
4	171319
5	17131925
6	1713192531
7	171319253137
8	17131925313743
9	1713192531374349
10	171319253137434955
11	$\boxed{17131925313743495561}$
12	1713192531374349556167
13	$\boxed{171319253137434955616773}$
14	17131925313743495561677379
15	1713192531374349556167737985
16	171319253137434955616773798591
17	17131925313743495561677379859197
18	17131925313743495561677379859197103
19	17131925313743495561677379859197103109
20	17131925313743495561677379859197103109115
21	17131925313743495561677379859197103109115121
22	17131925313743495561677379859197103109115121127
23	17131925313743495561677379859197103109115121127133
24	17131925313743495561677379859197103109115121127133139
25	17131925313743495561677379859197103109115121127133139145

where $\boxed{17}$, $\boxed{17131925313743495561}$ and $\boxed{171319253137434955616773}$ are primes.

Table 3.26: Back Concatenated $c6$ Sequence

k	$bc6_k$
1	1
2	$\boxed{71}$

Continued on next page

k	$bc6_k$
3	1371
4	191371
5	25191371
6	3125191371
7	373125191371
8	43373125191371
9	4943373125191371
10	554943373125191371
11	61554943373125191371
12	6761554943373125191371
13	736761554943373125191371
14	79736761554943373125191371
15	8579736761554943373125191371
16	918579736761554943373125191371
17	97918579736761554943373125191371
18	10397918579736761554943373125191371
19	10910397918579736761554943373125191371
20	11510910397918579736761554943373125191371
21	12111510910397918579736761554943373125191371
22	$\boxed{12712111510910397918579736761554943373125191371}$
23	13312712111510910397918579736761554943373125191371
24	13913312712111510910397918579736761554943373125191371
25	14513913312712111510910397918579736761554943373125191371

where $\boxed{71}$ and $\boxed{12712111510910397918579736761554943373125191371}$ are primes.

3.5.8 Concatenated Square Sequence

$L := 23$ $k := 1..L$ $sq_k := k^2$ $csq := ConS(sq, L)$.

Table 3.27: Concatenated Square Sequence

k	csq_k
1	1
2	14

Continued on next page

k	csq_k
3	$\boxed{149}$
4	14916
5	1491625
6	149162536
7	14916253649
8	1491625364964
9	149162536496481
10	149162536496481100
11	149162536496481100121
12	149162536496481100121144
13	149162536496481100121144169
14	149162536496481100121144169196
15	149162536496481100121144169196225
16	149162536496481100121144169196225256
17	149162536496481100121144169196225256289
18	149162536496481100121144169196225256289324
19	149162536496481100121144169196225256289324361
20	149162536496481100121144169196225256289324361400
21	149162536496481100121144169196225256289324361400441
22	149162536496481100121144169196225256289324361400441484
23	149162536496481100121144169196225256289324361400441484529

3.5.9 Back Concatenated Square Sequence

$L := 23$ $k := 1..L$ $sq_k := k^2$ $bcsq := BConS(sq, L)$.

Table 3.28: Back Concatenated Square Sequence

k	$bcsq_k$
1	1
2	$\boxed{41}$
3	$\boxed{941}$
4	16941
5	2516941
6	362516941
7	49362516941

Continued on next page

k	$bcsq_k$
8	6449362516941
9	816449362516941
10	100816449362516941
11	121100816449362516941
12	144121100816449362516941
13	169144121100816449362516941
14	$\boxed{196169144121100816449362516941}$
15	225196169144121100816449362516941
16	256225196169144121100816449362516941
17	289256225196169144121100816449362516941
18	324289256225196169144121100816449362516941
19	361324289256225196169144121100816449362516941
20	400361324289256225196169144121100816449362516941
21	441400361324289256225196169144121100816449362516941
22	484441400361324289256225196169144121100816449362516941
23	529484441400361324289256225196169144121100816449362516941

3.6 Permutation Sequence

Table 3.29: Permutation sequence

n
12
1342
135642
13578642
13579108642
135791112108642
1357911131412108642
13579111315161412108642
135791113151718161412108642
1357911131517192018161412108642
...

Questions:

1. Is there any perfect power among these numbers?

2. Their last digit should be: either 2 for exponents of the form $4k+1$, either 8 for exponents of the form $4k+3$, where $k \geq 0$?

3.7 Generalized Permutation Sequence

If $g : \mathbb{N}^* \to \mathbb{N}^*$, as a function, giving the number of digits of $a(n)$, and F is a permutation of $g(n)$ elements, then: $a(n) = F(1)F(2)\dots F(g(n))$.

3.8 Combinatorial Sequences

Combinations of 4 taken by 3 for the set $\{1,2,3,4\}$: 123, 124, 134, 234, combinations of 5 taken by 3 for the set $\{1,2,3,4,5\}$: 123, 124, 125, 134, 135, 145, 234, 235, 245, 345, combinations of 6 taken by 3 for the set $\{1,2,3,4,5,6\}$: 123, 124, 125, 126, 134, 135, 136, 145, 146, 156, 234, 235, 236, 245, 246, 256, 345, 346, 356, 456, ...

Which is the set of 5 decimal digits for which we have the biggest number of primes for numbers obtained by combinations of 5 – digits taken by 3?

We consider 5 digits of each in the numeration base 10, then, from mentioned sets, it results numbers by combinations of 5 – digits taken by 3 as it follows:

$\{7,9,5,3,1\} \Longrightarrow$

$n_{(10)}$	factors
795	$3\cdot5\cdot53$
793	$13\cdot61$
791	$7\cdot113$
753	$3\cdot251$
751	$\boxed{751}$
731	$17\cdot43$
953	$\boxed{953}$
951	$3\cdot317$
931	$7^2\cdot19$
531	$3^2\cdot59$

$\{9,7,5,3,1\} \Longrightarrow$

$n_{(10)}$	factors
975	$3\cdot5^2\cdot13$
973	$7\cdot139$
971	$\boxed{971}$
953	$\boxed{953}$
951	$3\cdot317$
931	$7^2\cdot19$
753	$3\cdot251$
751	$\boxed{751}$
731	$17\cdot43$
531	$3^2\cdot59$

$\{5,7,1,9,3\} \Longrightarrow$

$n_{(10)}$	factors
571	$\boxed{571}$
579	$3 \cdot 193$
573	$3 \cdot 191$
519	$3 \cdot 173$
513	$3^3 \cdot 19$
593	$\boxed{593}$
719	$\boxed{719}$
713	$23 \cdot 31$
793	$13 \cdot 61$
193	$\boxed{193}$

$\{3,1,5,7,9\} \Longrightarrow$

$n_{(10)}$	factors
315	$3^2 \cdot 5 \cdot 7$
317	$\boxed{317}$
319	$11 \cdot 29$
357	$3 \cdot 7 \cdot 17$
359	$\boxed{359}$
379	$\boxed{379}$
157	$\boxed{157}$
159	$3 \cdot 53$
179	$\boxed{179}$
579	$3 \cdot 193$

$\{1,3,5,7,9\} \Longrightarrow$

$n_{(10)}$	factors
135	$3^3 \cdot 5$
137	$\boxed{137}$
139	$\boxed{139}$
157	$\boxed{157}$
159	$3 \cdot 53$
179	$\boxed{179}$
357	$3 \cdot 7 \cdot 17$
359	$\boxed{359}$
379	$\boxed{379}$
579	$3 \cdot 193$

$\{1,4,3,7,9\} \Longrightarrow$

$n_{(10)}$	factors
143	$11 \cdot 13$
147	$3 \cdot 7^2$
149	$\boxed{149}$
137	$\boxed{137}$
139	$\boxed{139}$
179	$\boxed{179}$
437	$19 \cdot 23$
439	$\boxed{439}$
479	$\boxed{479}$
379	$\boxed{379}$

$\{1,3,0,7,9\} \Longrightarrow$

$n_{(10)}$	factors
130	$2 \cdot 5 \cdot 13$
137	$\boxed{137}$
139	$\boxed{139}$
107	$\boxed{107}$
109	$\boxed{109}$
179	$\boxed{179}$
307	$\boxed{307}$
309	$3 \cdot 103$
379	$\boxed{379}$
79	$\boxed{79}$

$\{1,0,3,7,9\} \Longrightarrow$

$n_{(10)}$	factors
103	$\boxed{103}$
107	$\boxed{107}$
109	$\boxed{109}$
137	$\boxed{137}$
139	$\boxed{139}$
179	$\boxed{179}$
37	$\boxed{37}$
39	$3 \cdot 13$
79	$\boxed{79}$
379	$\boxed{379}$

In conclusion, it was determined that the set 1, 0, 3, 7, 9 generates primes by combining 5 – digit taken by 3.

Generalization: which is the set of m – decimal digits for which we have the biggest number of primes for numbers obtained by combining m – digits taken by n?

$$\{2,4,0,6,8,5,1,3,7,9\} \Longrightarrow \begin{pmatrix} \vdots \\ \{2,0,6,8,3,9\} \\ \vdots \end{pmatrix} \Longrightarrow$$

$n_{(10)}$	factors
2068	$2^2 \cdot 11 \cdot 47$
2063	$\boxed{2063}$
2069	$\boxed{2069}$
2083	$\boxed{2083}$
2089	$\boxed{2089}$
2039	$\boxed{2039}$
2683	$\boxed{2683}$
2689	$\boxed{2689}$
2639	$7 \cdot 13 \cdot 29$
2839	$17 \cdot 167$
683	$\boxed{683}$
689	$13 \cdot 53$
639	$3^2 \cdot 71$
839	$\boxed{839}$
6839	$7 \cdot 977$

Of all $210 = C_{10}^6$ combinations of 10 – digits taken by 6, the most 4 – digit primes are the numbers for digits 2, 0, 6, 8, 3, 9. Of all $15 = C_6^4$ combinations of 6 – digits taken by 4 we have 9 primes.

3.9 Simple Numbers

Definition 3.3 ([Smarandache, 2006])**.** A number n is called *simple number* if the product of its proper divisors is less than or equal to n.

By analogy with the divisor function $\sigma_1(n)$, let

$$\Pi_k(n) = \sum_{d|n} d\ , \tag{3.1}$$

denote the product of the divisors d of n (including n itself). The function $\sigma_0 : \mathbb{N}^* \to \mathbb{N}^*$ counted the divisors of n. If $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_m^{\alpha_m}$, then $\sigma_0(n) = (\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_m + 1)$, [Weisstein, 2014a].

Theorem 3.4. *The divisor product* (3.1) *satisfies the identity*

$$\Pi(n) = n^{\frac{\sigma_0(n)}{2}}\ . \tag{3.2}$$

Proof. Let $n = p_1^{\alpha_1} p_2^{\alpha_2}$, where $p_1, p_2 \in \mathbb{P}_{\geq 2}$ si $n \in \mathbb{N}^*$, then divisors of n are:

$$\begin{array}{ccccc}
p_1^0 p_2^0\,, & p_1^0 p_2^1\,, & p_1^0 p_2^2\,, & \ldots, & p_1^0 p_2^{\alpha_2}\,,\\
p_1^1 p_2^0\,, & p_1^1 p_2^1\,, & p_1^1 p_2^2\,, & \ldots, & p_1^1 p_2^{\alpha_2}\,,\\
p_1^2 p_2^0\,, & p_1^2 p_2^1\,, & p_1^2 p_2^2\,, & \ldots, & p_1^2 p_2^{\alpha_2}\,,\\
\vdots & \vdots & \vdots & & \vdots\\
p_1^{\alpha_1} p_2^0\,, & p_1^{\alpha_1} p_2^1\,, & p_1^{\alpha_1} p_2^2\,, & \ldots, & p_1^{\alpha_1} p_2^{\alpha_2}\,.
\end{array}$$

The products lines of divisors are:

$$\begin{array}{rcl}
(p_1^0 p_2^0)\cdot(p_1^0 p_2^1)\cdot(p_1^0 p_2^2)\cdots(p_1^0 p_2^{\alpha_2}) & = & p_1^0 p_2^{\frac{\alpha 2(\alpha 2+1)}{2}}\,,\\
(p_1^1 p_2^0)\cdot(p_1^1 p_2^1)\cdot(p_1^1 p_2^2)\cdots(p_1^1 p_2^{\alpha_2}) & = & p_1^{1(\alpha_2+1)} p_2^{\frac{\alpha 2(\alpha 2+1)}{2}}\,,\\
(p_1^2 p_2^0)\cdot(p_1^2 p_2^1)\cdot(p_1^2 p_2^2)\cdots(p_1^2 p_2^{\alpha_2}) & = & p_1^{2(\alpha_2+1)} p_2^{\frac{\alpha 2(\alpha 2+1)}{2}}\,,\\
& \vdots & \\
(p_1^{\alpha_1} p_2^0)\cdot(p_1^{\alpha_1} p_2^1)\cdot(p_1^{\alpha_1} p_2^2)\cdots(p_1^{\alpha_1} p_2^{\alpha_2}) & = & p_1^{\alpha_1(\alpha_2+1)} p_2^{\frac{\alpha 2(\alpha 2+1)}{2}}\,.
\end{array}$$

Now we can write the product of all divisors:

$$\begin{aligned}
\Pi(n) = (p_1^0 p_2^0)\cdot(p_1^0 p_2^1)\cdots(p_1^{\alpha_1} p_2^2)\cdots(p_1^{\alpha_1} p_2^{\alpha_2}) & \\
= p_1^{\frac{\alpha_1(\alpha_1+1)}{2}(\alpha_2+1)} p_2^{(\alpha_1+1)\frac{\alpha_2(\alpha_2+1)}{2}} & \\
= p_1^{\frac{(\alpha_1+1)(\alpha_2+1)}{2}\alpha_1} p_2^{\frac{(\alpha_1+1)(\alpha_2+1)}{2}\alpha_2} & \\
= \left(p_1^{\alpha_1} p_2^{\alpha_2}\right)^{\frac{\sigma_0(n)}{2}} = n^{\frac{\sigma_0(n)}{2}}\,. &
\end{aligned}$$

By induction, it can be analogously proved the same identity for numbers that have the decomposition in m-prime factors $n = p_1^{\alpha_1}\cdot p_2^{\alpha_2}\cdots p_m^{\alpha_m}$. □

The table 3.30 gives values of n for which $\Pi(n)$ is a $m\ th$ power. Lionnet [1879] considered the case $m = 2$, [Weisstein, 2014a].

m	OEIS	n
2	[Sloane, 2014, A048943]	1, 6, 8, 10, 14, 15, 16, 21, 22, 24, 26, ...
3	[Sloane, 2014, A048944]	1, 4, 8, 9, 12, 18, 20, 25, 27, 28, 32, ...
4	[Sloane, 2014, A048945]	1, 24, 30, 40, 42, 54, 56, 66, 70, 78, ...
5	[Sloane, 2014, A048946]	1, 16, 32, 48, 80, 81, 112, 144, 162, ...

Table 3.30: Table which $\Pi(n)$ is a m–th power

For $n = 1, 2, \ldots$ and $k = 1$ the first few values are 1, 2, 3, 8, 5, 36, 7, 64, 27, 100, 11, 1728, 13, 196, ..., [Sloane, 2014, A007955].

Likewise, we can define the function

$$P(n) = \sum_{d|n} d \ , \tag{3.3}$$

denoting the product of the proper divisors d of n. Then definition 3.3 becomes

Definition 3.5. The number $n \in \mathbb{N}$ is *simple number* if and only if $P(n) \le n$.

We have a similar identity with (3.2), [Lucas, 1891, Ex. VI, p. 373].

$$P(n) = n^{\frac{\sigma_0(n)}{2}-1} \ . \tag{3.4}$$

Theorem 3.6. *The product of proper divisors satisfies the identity* (3.4).

Proof. Let $n = p_1^{\alpha_1} p_2^{\alpha_2}$, where $p_1, p_2 \in \mathbb{P}_{\ge 2}$ si $n \in \mathbb{N}^*$, then proper divisors of n are:

$$\begin{array}{cccccc}
 & p_1^0 p_2^1\,, & p_1^0 p_2^2\,, & \ldots, & p_1^0 p_2^{\alpha_2-1}\,, & p_1^0 p_2^{\alpha_2}\,,\\
p_1^1 p_2^0\,, & p_1^1 p_2^1\,, & p_1^1 p_2^2\,, & \ldots, & p_1^1 p_2^{\alpha_2-1}\,, & p_1^1 p_2^{\alpha_2}\,,\\
p_1^2 p_2^0\,, & p_1^2 p_2^1\,, & p_1^2 p_2^2\,, & \ldots, & p_1^2 p_2^{\alpha_2-1}\,, & p_1^2 p_2^{\alpha_2}\,,\\
\vdots & \vdots & \vdots & & \vdots & \\
p_1^{\alpha_1} p_2^0\,, & p_1^{\alpha_1} p_2^1\,, & p_1^{\alpha_1} p_2^2\,, & \ldots, & p_1^{\alpha_1} p_2^{\alpha_2-1}\,. &
\end{array}$$

The products lines of divisors are:

$$\begin{array}{rcl}
(p_1^0 p_2^1)\cdot(p_1^0 p_2^2)\cdots(p_1^0 p_2^{\alpha_2-1})\cdot(p_1^0 p_2^{\alpha_2}) & = & p_1^0 p_2^{\frac{\alpha 2(\alpha 2+1)}{2}}\\
(p_1^1 p_2^0)\cdot(p_1^1 p_2^1)\cdot(p_1^1 p_2^2)\cdots(p_1^1 p_2^{\alpha_2-1})\cdot(p_1^1 p_2^{\alpha_2}) & = & p_1^{1(\alpha_2+1)} p_2^{\frac{\alpha 2(\alpha 2+1)}{2}}\\
(p_1^2 p_2^0)\cdot(p_1^2 p_2^1)\cdot(p_1^2 p_2^2)\cdots(p_1^2 p_2^{\alpha_2-1})\cdot(p_1^2 p_2^{\alpha_2}) & = & p_1^{2(\alpha_2+1)} p_2^{\frac{\alpha 2(\alpha 2+1)}{2}}\\
 & \vdots & \\
(p_1^{\alpha_1} p_2^0)\cdot(p_1^{\alpha_1} p_2^1)\cdot(p_1^{\alpha_1} p_2^2)\cdots(p_1^{\alpha_1} p_2^{\alpha_2-1}) & = & \dfrac{p_1^{\alpha_1(\alpha_2+1)}}{p_1^{\alpha_1}}\cdot\dfrac{p_2^{\frac{\alpha 2(\alpha 2+1)}{2}}}{p_2^{\alpha_2}}
\end{array}$$

Now we can write the product of all proper divisors:

$$\begin{aligned}
P(n) &= (p_1^0 p_2^1)\cdots(p_1^{\alpha_1} p_2^2)\cdots(p_1^{\alpha_1} p_2^{\alpha_2-1})\\
&= \frac{p_1^{\frac{\alpha_1(\alpha_1+1)}{2}(\alpha_2+1)}}{p_1^{\alpha_1}}\,\frac{p_2^{(\alpha_1+1)\frac{\alpha_2(\alpha_2+1)}{2}}}{p_2^{\alpha_2}} = p_1^{\frac{\sigma_0(n)}{2}\alpha_1-\alpha_1}\,p_2^{\frac{\sigma_0(n)}{2}\alpha_2-\alpha_2}\\
&= \left(p_1^{\alpha_1} p_2^{\alpha_2}\right)^{\frac{\sigma_0(n)}{2}-1} = n^{\frac{\sigma_0(n)}{2}-1} \ .
\end{aligned}$$

By induction, it can be analogously proved the same identity for numbers that have the decomposition in m – prime factors $n = p_1^{\alpha_1}\cdot p_2^{\alpha_2}\cdots p_m^{\alpha_m}$. □

Observation 3.7. To note that $P(n) = 1$ if and only if $n \in \mathbb{P}_{\geq 2}$.

Observation 3.8. Due to Theorem *3.6* one can give the following definition of simple numbers: Any natural number that has more than 2 divisors by its own is a *simple number*. Obviously, we can say that any natural number that has more than 2 divisors by its own is a *non-simple (complex) number*.

Using the function 3.3 one can write a simple program for determining the simple numbers: 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 15, 17, 19, 21, 22, 23, 25, 26, 27, 29, 31, 33, 34, 35, 37, 38, 39, 41, 43, 46, 47, 49, 51, 53, 55, 57, 58, 59, 61, 62, 65, 67, 69, 71, 73, 74, 77, 79, 82, 83, 85, 86, 87, 89, 91, 93, 94, 95, 97, 101, 103, 106, 107, 109, 111, 113, 115, 118, 119, 121, 122, 123, 125, 127, 129, 131, 133, 134, 137, 139, 141, 142, 143, 145, 146, 149, ...
or the non–simple (complex) numbers: 12, 16, 18, 20, 24, 28, 30, 32, 36, 40, 42, 44, 45, 48, 50, 52, 54, 56, 60, 63, 64, 66, 68, 70, 72, 75, 76, 78, 80, 81, 84, 88, 90, 92, 96, 98, 99, 100, 102, 104, 105, 108, 110, 112, 114, 116, 117, 120, 124, 126, 128, 130, 132, 135, 136, 138, 140, 144, 147, 148, 150, ...

How many simple numbers or non–simple (complex) numbers we have to the limit L=100, 1000, 10000, 20000, 30000, 40000, 50000, ...? The Table 3.31 answers the question:

L	Simple	Complex
100	61	38
1000	471	528
10000	3862	6137
20000	7352	12647
30000	10717	17282
40000	14004	25995
50000	17254	32745

Table 3.31: How many simple numbers or non–simple

3.10 Pseudo–Smarandache Numbers

Let n be a positive natural number.

Definition 3.9. The *pseudo–Smarandache* number of order o ($o = 1, 2, \ldots$) of $n \in \mathbb{N}^*$ is the first natural number m for which

$$S_o(m) = 1^o + 2^o + \ldots + m^o \tag{3.5}$$

divides to n. The *pseudo–Smarandache* number of first kind is simply called *pseudo–Smarandache* number.

3.10.1 Pseudo–Smarandache Numbers of First Kind

The *pseudo–Smarandache* number of first kind to $L=50$ sunt: 1, 3, 2, 7, 4, 3, 6, 15, 8, 4, 10, 8, 12, 7, 5, 31, 16, 8, 18, 15, 6, 11, 22, 15, 24, 12, 26, 7, 28, 15, 30, 63, 11, 16, 14, 8, 36, 19, 12, 15, 40, 20, 42, 32, 9, 23, 46, 32, 48, 24 , obtained by calling the function Z_1 given by the program 2.109, $n := 1..L$, $Z_1(n) =$.

Definition 3.10. The function $Z_1^k(n) = Z_1(Z_1(\ldots(Z_1(n))))$ is defined, where composition repeats k–times.

We present a list of issues related to the function Z_1, with total or partial solutions.

1. Is the series
$$\sum_{n=1}^{\infty} \frac{1}{Z_1(n)} \tag{3.6}$$
convergent?

   Because
$$\sum_{n=1}^{\infty} \frac{1}{Z_1(n)} \geq \sum_{n=1}^{\infty} \frac{1}{2n-1} = \infty\,,$$
it follows that the series (3.6) is divergent.

2. Is the series
$$\sum_{n=1}^{\infty} \frac{Z_1(n)}{n} \tag{3.7}$$
convergent?

   Because
$$\sum_{n=1}^{\infty} \frac{Z_1(n)}{n} \geq \sum_{n=1}^{\infty} \frac{\sqrt{8n+1}-1}{2n} > \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}\,,$$
and because the harmonic series
$$\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}}$$
with $0 < \alpha \leq 1$ is divergent, according to [Acu, 2006], then it follows that the series (3.7) is divergent.

3. Is the series
$$\sum_{n=1}^{\infty} \frac{Z_1(n)}{n^2} \tag{3.8}$$
convergent?

4. For a given pair of integers $k, m \in \mathbb{N}^*$, find all integers n such that $Z_1^k(n) = m$. How many solutions are there?

   Program 3.11. for determining the solutions of Diophantine equations $Z_1^2(n) = m$.

$$
P2Z1(n_a, n_b, m_a, m_b) := \left|
\begin{array}{l}
k \leftarrow 1 \\
for\ m \in m_a..m_b \\
\quad \left|
\begin{array}{l}
j \leftarrow 2 \\
for\ n \in n_a..n_b \\
\quad if\ m{=}Z_1(Z_1(n)) \\
\quad\quad \left|
\begin{array}{l}
S_{k,j} \leftarrow n \\
j \leftarrow j+1
\end{array}
\right. \\
if\ j > 3 \\
\quad \left|
\begin{array}{l}
S_{k,1} \leftarrow m \\
k \leftarrow k+1
\end{array}
\right.
\end{array}
\right. \\
return\ S
\end{array}
\right.
$$

   Using the program 3.11 we can determine all the solutions, for any $n \in \{n_a, n_a+1, \ldots, n_b\}$ and any $m \in \{m_a, m_a+1, \ldots, m_b\}$, where m is the right part of the equation $Z_1^2(n){=}m$. For example for $n \in \{20, 21, \ldots, 100\}$ and $m \in \{12, 13, \ldots, 22\}$ we give the solutions in Table 3.32.

m	n
12	27, 52, 79, 91;
14	90;
15	25, 31, 36, 41, 42, 50, 61, 70, 75, 82, 93, 100;
16	51, 85;
18	38, 95;
20	43, 71;
22	46, 69, 92;

Table 3.32: The solutions of Diophantine equations $Z_1^2(n) = m$

   Using a similar program (instead of condition $m{=}Z_1(Z_1(n))$ one puts the condition $m{=}Z_1(Z_1(Z_1(n)))$) we can obtain all the solutions for the equation $Z_1^3(n) = m$ with $n \in \{20, 21, \ldots, 100\}$ and $m \in \{12, 13, \ldots, 22\}$:

5. Are the following values bounded or unbounded

   (a) $|Z_1(n+1) - Z_1(n)|$;

m	n
12	53, 54, 63;
15	26, 37, 39, 43, 45, 57, 62, 65, 71, 74, 78, 83;
20	86;
22	47;

Table 3.33: The solutions of Diophantine equations $Z_1^3(n) = m$

(b) $Z_1(n+1)/Z_1(n)$.

We have the inequalities:

$$|Z_1(n+1) - Z_1(n)| \le \left|2n+1-\left\lceil\frac{\sqrt{8n+1}-1}{2}\right\rceil\right|,$$

$$\frac{\lceil s_1(n+1)\rceil}{2n-1} \le \frac{Z_1(n+1)}{Z_1(n)} \le \frac{2n+1}{\lceil s_1(n)\rceil}.$$

6. Find all values of n such that:

   (a) $Z_1(n) = Z_1(n+1)$, for $n \in \{1,2,\ldots,10^5\}$ there is no n, to verify the equality;

   (b) $Z_1(n) \mid Z_1(n+1)$, for $n \in \{1,2,\ldots,10^3\}$ obtain: 1, 6, 22, 28, 30, 46, 60, 66, 102, 120, 124, 138, 156, 166, 190, 262, 276, 282, 316, 348, 358, 378, 382, 399, 430, 478, 486, 498, 502, 508, 606, 630, 642, 700, 718, 732, 742, 750, 760, 786, 796, 822, 828, 838, 852, 858, 862, 886, 946, 979, 982;

   (c) $Z_1(n+1) \mid Z_1(n)$, for $n \in \{1,2,\ldots,10^3\}$ obtain: 9, 17, 25, 41, 49, 73, 81, 97, 113, 121, 169, 193, 233, 241, 257, 313, 337, 361, 401, 433, 457, 577, 593, 601, 617, 625, 673, 761, 841, 881, 977;

7. Is there an algorithm that can be used to solve each of the following equations?

   (a) $Z_1(n) + Z_1(n+1) = Z_1(n+2)$ with the solutions: 609, 696, for $n \in \{1,2,\ldots,10^3\}$;

   (b) $Z_1(n) = Z_1(n+1) + Z_1(n+2)$ with the solutions: 4, 13, 44, 83, 491, for $n \in \{1,2,\ldots,10^3\}$;

   (c) $Z_1(n) \cdot Z_1(n+1) = Z_1(n+2)$ not are solutions, for $n \in \{1,2,\ldots,10^3\}$;

   (d) $2 \cdot Z_1(n+1) = Z_1(n) + Z_1(n+2)$ not are solutions, for $n \in \{1,2,\ldots,10^3\}$;

   (e) $Z_1(n+1)^2 = Z_1(n) \cdot Z_1(n+2)$ not are solutions, for $n \in \{1,2,\ldots,10^3\}$.

8. There exists $n \in \mathbb{N}^*$ such that:

   (a) $Z_1(n) < Z_1(n+1) < Z_1(n+2) < Z_1(n+3)$? The following numbers, for $n \le 10^3$, have the required propriety: 91, 159, 160, 164, 176, 224, 248, 260, 266, 308, 380, 406, 425, 469, 483, 484, 496, 551, 581, 590, 666, 695, 754, 790, 791, 805, 806, 812, 836, 903, 904. There are infinite instances of 3 consecutive increasing terms in this sequence?

   (b) $Z_1(n) > Z_1(n+1) > Z_1(n+2) > Z_1(n+3)$? Up to the limit $L = 10^3$ we have the numbers: 97, 121, 142, 173, 214, 218, 219, 256, 257, 289, 302, 361, 373, 421, 422, 439, 529, 577, 578, 607, 669, 673, 686, 712, 751, 757, 761, 762, 773, 787, 802, 890, 907, 947 which verifies the required condition. There are infinite instances of 3 consecutive decreasing terms in this sequence?

   (c) $Z_1(n) > Z_1(n+1) > Z_1(n+2) > Z_1(n+3) > Z_1(n+4)$? Up to $n \le 10^3$ there are the numbers: 159, 483, 790, 805, 903, which verifies the required condition. There are infinite instances of 3 consecutive decreasing terms in this sequence?

   (d) $Z_1(n) < Z_1(n+1) < Z_1(n+2) < Z_1(n+3) < Z_1(n+4)$? Up to $n \le 10^3$ there are the numbers: 218, 256, 421, 577, 761, which verifies the required condition. There are infinite instances of 3 consecutive decreasing terms in this sequence?

9. We denote by S, the *Smarandache function*, the function that attaches to any $n \in \mathbb{N}^*$ the smallest natural number m for which $m!$ is a multiple of n, [Smarandache, 1980, Cira and Smarandache, 2014]. The question arises whether there are solutions to the equations:

   (a) $Z_1(n) = S(n)$? In general, if $Z_1(n) = S(n) = m$, then $n \mid [m(m+1)/2]$ and $n \mid m!$ must be satisfied. So, in such cases, m is sometimes the biggest prime factor of n, although that is not always the case. For $n \le 10^2$ we have 19 solutions: 1, 6, 14, 15, 22, 28, 33, 38, 46, 51, 62, 66, 69, 86, 87, 91, 92, 94, 95 . There are an infinite number of such solutions?

   (b) $Z_1(n) = S(n) - 1$? Let $p \in \mathbb{P}_{\ge 5}$. Since it is well-known that $S(p) = p$, for $p \in \mathbb{P}_{\ge 5}$, it follows from a previous result that $Z_1(p) + 1 = S(p)$. Of course, it is likely that other solutions may exist. Up to 10^2 we have 37 solutions: 3, 5, 7, 10, 11, 13, 17, 19, 21, 23, 26, 29, 31, 34, 37, 39, 41, 43, 47, 53, 55, 57, 58, 59, 61, 67, 68, 71, 73, 74, 78, 79, 82, 83, 89, 93, 97 . Let us observe that there exists also primes as solutions, like: 10, 21, 26, 34, 39, 55, 57, 58, 68, 74, 78, 82, 93. There are an infinite number of such solutions?

(c) $Z_1(n) = 2\cdot S(n)$? This equation up to 10^3 has 33 solutions: 12, 35, 85, 105, 117, 119, 185, 217, 235, 247, 279, 335, 351, 413, 485, 511, 535, 555, 595, 603, 635, 651, 685, 707, 741, 781, 785, 835, 893, 923, 925, 927, 985. In general, there are solutions for equation $Z_1(n) = k\cdot S(n)$, for $k = 2,3,\dots,16$ and $n \le 10^3$. There are an infinite number of such solutions?

3.10.2 Pseudo–Smarandache Numbers of Second Kind

Pseudo–Smarandache numbers of second kind up to $L = 50$ are: 1, 3, 4, 7, 2, 4, 3, 15, 13, 4, 5, 8, 6, 3, 4, 31, 8, 27, 9, 7, 13, 11, 11, 31, 12, 12, 40, 7, 14, 4, 15, 63, 22, 8, 7, 40, 18, 19, 13, 15, 20, 27, 21, 16, 27, 11, 23, 31, 24, 12 , obtained by calling the function Z_2 given by the program 2.113, $n := 1..L$, $Z_2(n) =$.

Definition 3.12. The function $Z_2^k(n) = Z_2(Z_2(\dots(Z_2(n))))$ is defined by the composition which repeats of k times.

We present a list, similar to that of the function Z_1, of issues related to the function Z_2, with total or partial solutions.

1. Is the series
$$\sum_{n=1}^{\infty} \frac{1}{Z_2(n)} \tag{3.9}$$
convergent?

   Because
$$\sum_{n=1}^{\infty} \frac{1}{Z_2(n)} \ge \sum_{n=1}^{\infty} \frac{1}{2n-1} = \infty\,,$$
it follows that the series (3.9) is divergent.

2. Is the series
$$\sum_{n=1}^{\infty} \frac{Z_2(n)}{n} \tag{3.10}$$
convergent?

   Because
$$\sum_{n=1}^{\infty} \frac{Z_2(n)}{n} \ge \sum_{n=1}^{\infty} \frac{s_2(n)}{n} > \sum_{n=1}^{\infty} \frac{1}{n^{\frac{2}{3}}} = \infty\,,$$
where $s_2(n)$ is given by (2.103), and for harmonic series
$$\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}}$$
with $0 < \alpha \le 1$ is divergent, according to Acu [2006], then it follows that the series (3.10) is divergent.

3. Is the series
$$\sum_{n=1}^{\infty} \frac{Z_2(n)}{n^2} \tag{3.11}$$
convergent?

4. For a given pair of integers $k, m \in \mathbb{N}^*$, find all integers n such that $Z_2^k(n) = m$. How many solutions are there?

   Using a similar program with the program 3.11 in which, instead of the condition $m = Z_1(Z_1(n))$ we put the condition $m = Z_2(Z_2(n))$. With this program we can determine all the solutions for any $n \in \{n_a, n_a+1, \ldots, n_b\}$ and any $m \in \{m_a, m_a+1, \ldots, m_b\}$, where m is the right part of equation $Z_2^2(n) = m$. For example, for $n \in \{20, 21, \ldots, 100\}$ and $m \in \{12, 13, \ldots, 22\}$ we give the solution in the Table 3.34.

m	n
12	53;
13	32, 43, 57, 79, 95, 96;
14	59;
15	24, 27, 34, 36, 48, 51, 54, 56, 60, 68, 84, 93;
16	89;
20	83;
21	86;
22	67;

.

Table 3.34: The solutions of Diophantine equations $Z_2^2(n) = m$

   With a similar program (instead of condition $m = Z_2(Z_2(n))$ one puts the condition $m=Z_2(Z_2(Z_2(n)))$) one can obtain all solutions for equation $Z_2^3(n) = m$ with $n \in \{20, 21, \ldots, 100\}$ and $m \in \{12, 13, \ldots, 22\}$:

m	n
13	38, 52, 64, 80, 86;
15	25, 26, 42, 44, 45, 49, 50, 65, 66, 73, 74, 85, 88, 90, 97, 98, 99, 100;

Table 3.35: The solutions for equation $Z_2^3(n) = m$

5. Are the following values bounded or unbounded

   (a) $|Z_2(n+1) - Z_2(n)|$;

   (b) $Z_2(n+1)/Z_2(n)$.

We have the inequalities:

$$|Z_2(n+1)-Z_2(n)| \leq |2n+1)-\lceil s_2(n)\rceil| \ ,$$

$$\frac{\lceil s_2(n+1)\rceil}{2n-1} \leq \frac{Z_2(n+1)}{Z_2(n)} \leq \frac{2n+1}{\lceil s_2(n)\rceil} \ .$$

6. Find all values of n such that:

   (a) $Z_2(n) = Z_2(n+1)$, for $n \in \{1,2,\ldots,5\cdot 10^2\}$ we obtained 10 solutions: 22, 25, 73, 121, 166, 262, 313, 358, 361, 457;

   (b) $Z_2(n) \mid Z_2(n+1)$, for $n \in \{1,2,\ldots,5\cdot 10^2\}$ we obtained 21 solutions: 1, 5, 7, 22, 25, 28, 51, 70, 73, 95, 121, 143, 166, 190, 262, 313, 358, 361, 372, 457, 473;

   (c) $Z_2(n+1) \mid Z_2(n)$, for $n \in \{1,2,\ldots,5\cdot 10^2\}$ we obtained 28 solutions: 13, 18, 22, 25, 49, 54, 61, 73, 97, 109, 121, 128, 157, 162, 166, 174, 193, 218, 241, 262, 289, 313, 337, 358, 361, 368, 397, 457;

7. Is there an algorithm that can be used to solve each of the following equations?

   (a) $Z_2(n)+Z_2(n+1) = Z_2(n+2)$ with the solutions: 1, 2, for $n \leq 10^3$;

   (b) $Z_2(n) = Z_2(n+1)+Z_2(n+2)$ with the solutions: 78, 116, 582, for $n \leq 10^3$;

   (c) $Z_2(n)\cdot Z_2(n+1) = Z_2(n+2)$ not are solutions, for $n \in\leq 10^3$;

   (d) $2\cdot Z_2(n+1) = Z_2(n)+Z_2(n+2)$ with the solution: 495, for $n \leq 10^3$;

   (e) $Z_2(n+1)^2 = Z_2(n)\cdot Z_2(n+2)$ not are solutions, for $n \leq 10^3$.

8. There is $n \in \mathbb{N}^*$ such that:

   (a) $Z_2(n) < Z_2(n+1) < Z_2(n+2) < Z_2(n+3)$? The following 24 numbers, for $n \leq 5\cdot 10^2$, have the required propriety: 1, 39, 57, 111, 145, 146, 147, 204, 275, 295, 315, 376, 380, 381, 391, 402, 406, 425, 445, 477, 494, 495, 496, 497. There are infinite instances of 3 consecutive increasing terms in this sequence?

   (b) $Z_2(n) > Z_2(n+1) > Z_2(n+2) > Z_2(n+3)$? Up to the limit $L = 5\cdot 10^2$ we have 21 numbers: 32, 48, 60, 162, 183, 184, 192, 193, 218, 228, 256, 257, 282, 332, 333, 342, 362, 422, 448, 449, 467, which verifies the required condition. There are infinite instances of 3 consecutive decreasing terms in this sequence?

(c) $Z_2(n) > Z_2(n+1) > Z_2(n+2) > Z_2(n+3) > Z_2(n+4)$? Up to $n \leq 10^3$ there are the numbers: 145, 146, 380, 494, 495, 496, 610, 805, 860, 930, 994, 995,which verifies the required condition. There are infinite instances of 3 consecutive increasing terms in this sequence?

(d) $Z_2(n) < Z_2(n+1) < Z_2(n+2) < Z_2(n+3) < Z_2(n+4)$? Up to $n \leq 10^3$ there are the numbers: 183, 192, 256, 332, 448, 547, 750, 751, which verifies the required condition. There are infinite instances of 3 consecutive increasing terms in this sequence?

9. We denote by S, the *Smarandache function*, i.e. the function which attaches to any $n \in \mathbb{N}^*$ the smallest natural number m for which $m!$ is a multiple of n, [Smarandache, 1980, Cira and Smarandache, 2014]. The question arises whether there are solutions to the equations:

   (a) $Z_2(n) = S(n)$? In general, if $Z_2(n) = S(n) = m$, then $n \mid [m(m+1)(2m+1)/6]$ and $n \mid m!$ must be satisfied. So, in such cases, m is sometimes the biggest prime factor of n, although that is not always the case. For $n \leq 3 \cdot 10^2$ we have 30 solutions: 1, 22, 28, 35, 38, 39, 70, 85, 86, 92, 93, 117, 118, 119, 134, 140, 166, 185, 186, 190, 201, 214, 217, 235, 247, 255, 262, 273, 278, 284 . There are an infinite number of such solutions?

   (b) $Z_2(n) = S(n) - 1$? Let $p \in \mathbb{P}_{\geq 5}$. Since it is well–known that $S(p) = p$, for $p \in \mathbb{P}_{\geq 5}$, it follows from a previous result that $Z_1(p) + 1 = S(p)$. Of course, it is likely that other solutions may exist. Up to $3 \cdot 10^2$ we have 26 solutions: 10, 15, 26, 30, 58, 65, 69, 74, 77, 106, 115, 122, 123, 130, 136, 164, 177, 187, 202, 215, 218, 222, 246, 265, 292, 298. There are an infinite number of such solutions?

   (c) $Z_2(n) = 2 \cdot S(n)$? This equation up to $2 \cdot 10^2$ has 7 solutions: 12, 33, 87, 141, 165, 209, 249. In general, there exists solutions for equation $Z_2(n) = k \cdot S(n)$, for $k = 2, 3, 4, 5$ and $n \leq 10^3$. There are an infinite number of such solutions?

3.10.3 Pseudo–Smarandache Numbers of Third Kind

The *pseudo–Smarandache* numbers of third rank up to $L = 50$ are: 1, 3, 2, 3, 4, 3, 6, 7, 2, 4, 10, 3, 12, 7, 5, 7, 16, 3, 18, 4, 6, 11, 22, 8, 4, 12, 8, 7, 28, 15, 30, 15, 11, 16, 14, 3, 36, 19, 12, 15, 40, 20, 42, 11, 5, 23, 46, 8, 6, 4 , obtained by calling the function given by the program 2.120, $n := 1..L$, $Z_3(n) =$.

Definition 3.13. In the function $Z_3^k(n) = Z_3(Z_3(\ldots(Z_3(n))))$ the composition repeats k times.

We present a list, similar to that of function Z_1, with issues concerning the function Z_3, with total or partial solutions.

1. Is the series
$$\sum_{n=1}^{\infty} \frac{1}{Z_3(n)} \tag{3.12}$$
convergent?

   Because
$$\sum_{n=1}^{\infty} \frac{1}{Z_3(n)} \geq \sum_{n=1}^{\infty} \frac{1}{n-1} = \infty\,,$$
it follows that the series (3.12) is divergent.

2. Is the series
$$\sum_{n=1}^{\infty} \frac{Z_3(n)}{n} \tag{3.13}$$
convergent?

   Because
$$\sum_{n=1}^{\infty} \frac{Z_3(n)}{n} \geq \sum_{n=1}^{\infty} \frac{s_3(n)}{n} > \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{4}}} = \infty\,,$$
therefore it follows that the series (3.13) is divergent (see Figure 3.1).

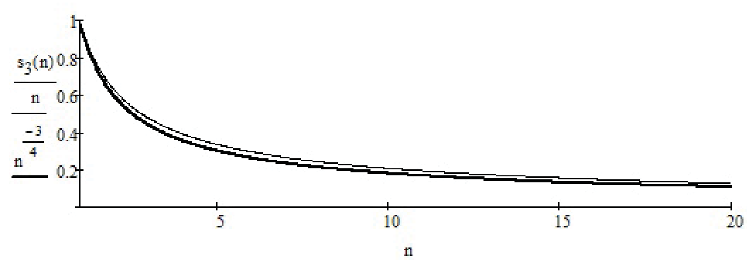

Figure 3.1: The functions $s_3(n)\cdot n^{-1}$ and $n^{-\frac{3}{4}}$

3. Is series
$$\sum_{n=1}^{\infty} \frac{Z_3(n)}{n^2} \tag{3.14}$$
convergent?

4. For a given pair of integers $k, m \in \mathbb{N}^*$, find all integers n such that $Z_3^k(n) = m$. How many solutions are there?

   Using a program similar to the program 3.11 where instead of condition $m = Z_1(Z_1(n))$ we put the condition $m = Z_3(Z_3(n))$. Using this program,

we can determine all solutions, or any $n \in \{n_a, n_a+1, \ldots, n_b\}$ and any $m \in \{m_a, m_a+1, \ldots, m_b\}$, where m is the right member of the equation $Z_3^2(n) = m$. For example, for $n \in \{20, 21, \ldots, 100\}$ and $m \in \{12, 13, \ldots, 22\}$ we have the solutions in Table 3.36:

m	n
12	53, 79, 91;
15	31, 41, 61, 82, 88, 93, 97;
16	51, 85;
18	38, 76, 95;
20	43, 71;
22	46, 69, 92;

.

Table 3.36: The solutions of Diophantine equations $Z_3^2(n) = m$

Using a similar program $\big($instead of condition $m = Z_3(Z_3(n))$ we put the condition m=$Z_3(Z_3(Z_3(n)))\big)$ we can obtain all solutions for the equation $Z_3^3(n) = m$ with $n \in \{20, 21, \ldots, 100\}$ and $m \in \{12, 13, \ldots, 22\}$:

m	n
15	62, 83, 89;
20	86;
22	47;

Table 3.37: The solutions of Diophantine equations $Z_3^3(n) = m$

5. Are the following values bounded or unbounded

   (a) $|Z_3(n+1) - Z_3(n)|$;

   (b) $Z_3(n+1)/Z_3(n)$.

   We have the inequalites:

$$|Z_3(n+1) - Z_3(n)| \le \left|(n+1)^2 - s_3(n)\right| \ ,$$

$$\frac{s_3(n+1)}{n^2} \le \frac{Z_3(n+1)}{Z_3(n)} \le \frac{(n+1)^2}{s_3(n)} \ .$$

6. Find all values of n such that:

   (a) $Z_3(n) = Z_3(n+1)$, for $n \in \left\{1, 2, \ldots, 10^3\right\}$ does not have solutions;

(b) $Z_3(n) \mid Z_3(n+1)$, for $n \in \{1,2,\ldots,3\cdot 10^2\}$ we obtained 35 solutions: 1, 6, 9, 12, 18, 22, 25, 28, 30, 36, 46, 60, 66, 72, 81, 100, 102, 112, 121, 138, 147, 150, 156, 166, 169, 172, 180, 190, 196, 198, 240, 262, 268, 276, 282;

(c) $Z_3(n+1) \mid Z_3(n)$, for $n \in \{1,2,\ldots,10^3\}$ we obtained 25 solutions: 24, 31, 41, 73, 113, 146, 168, 193, 257, 313, 323, 337, 401, 433, 457, 506, 575, 577, 601, 617, 673, 728, 761, 881, 977;

7. Is there an algorithm that can be used to solve each of the following equations?

   (a) $Z_3(n)+Z_3(n+1)=Z_3(n+2)$ with the solutions: 24, 132, 609, 979, for $n \in \{1,2,\ldots,10^3\}$;

   (b) $Z_3(n) = Z_3(n+1) + Z_3(n+2)$ with the solution: 13, for $n \in \{1,2,\ldots,10^3\}$;

   (c) $Z_3(n)\cdot Z_3(n+1)=Z_3(n+2)$ not are solutions, for $n \in \{1,2,\ldots,10^3\}$;

   (d) $2\cdot Z_3(n+1)=Z_3(n)+Z_3(n+2)$ with the solution: 3, 48, 318, 350, for $n \in \{1,2,\ldots,10^3\}$;

   (e) $Z_3(n+1)^2=Z_3(n)\cdot Z_3(n+2)$ not are solutions, for $n \in \{1,2,\ldots,10^3\}$.

8. There exists $n \in \mathbb{N}^*$ such that:

   (a) $Z_3(n)<Z_3(n+1)<Z_3(n+2)<Z_3(n+3)$? The next 25 number, for$n \le 8\cdot 10^2$, have the required property: 20, 56, 91, 164, 175, 176, 236, 308, 350, 380, 405, 406, 468, 469, 496, 500, 644, 650, 656, 666, 679, 680, 715, 716, 775. Are there infinitely many instances of 3 consecutive increasing terms in this sequence?

   (b) $Z_3(n)>Z_3(n+1)>Z_3(n+2)>Z_3(n+3)$? Up to the limit $L=8\cdot 10^2$ we have 21 numbers: 47, 109, 113, 114, 118, 122, 123, 157, 181, 193, 257, 258, 317, 397, 401, 402, 487, 526, 534, 541, 547, 613, 622, 634, 669, 701, 723, 757, 761, 762, which verifies the required condition. Are there infinitely many instances of 3 consecutive decreasing terms in this sequence?

   (c) $Z_3(n)>Z_3(n+1)>Z_3(n+2)>Z_3(n+3)>Z_3(n+4)$? Up to $n \le 10^3$ there are the numbers: 175, 405, 468, 679, 715, 805, 903, which verifies the required condition. Are there infinitely many instances of 3 consecutive increasing terms in this sequence?

   (d) $Z_3(n)<Z_3(n+1)<Z_3(n+2)<Z_3(n+3)<Z_3(n+4)$? Up to $n \le 10^3$ there are the numbers: 113, 122, 257, 401, 761, 829, which verifies

the required condition. Are there infinitely many instances of 3 consecutive increasing terms in this sequence?

9. We denote by S, the *Smarandache function*, i.e. the function that attach to any $n \in \mathbb{N}^*$ the smallest natural number m for which $m!$ is a multiple of n, [Smarandache, 1980, Cira and Smarandache, 2014]. The question arises whether there are solutions to the equations:

   (a) $Z_3(n) = S(n)$? In general, if $Z_3(n) = S(n) = m$, then $n \mid [m(m+1)/2]$ and $n \mid m!$ must be satisfied. So, in such cases, m is sometimes the biggest prime factor of n, although that is not always the case. For $n \le 10^2$ we have 23 solutions: 1, 6, 14, 15, 22, 28, 33, 38, 44, 46, 51, 56, 62, 66, 69, 76, 86, 87, 91, 92, 94, 95, 99 . There are an infinite number of such solutions?

   (b) $Z_3(n) = S(n) - 1$? Let $p \in \mathbb{P}_{\ge 5}$. Since it is well-known that $S(p) = p$, for $p \in \mathbb{P}_{\ge 5}$, it follows from a previous result that $Z_3(p) + 1 = S(p)$. Of course, it is likely that other solutions may exist. Up to 10^2 we have 46 solutions: 3, 4, 5, 7, 10, 11, 12, 13, 17, 19, 20, 21, 23, 26, 27, 29, 31, 34, 37, 39, 41, 43, 45, 47, 52, 53, 54, 55, 57, 58, 59, 61, 63, 67, 68, 71, 73, 74, 78, 79, 81, 82, 83, 89, 93, 97. There are an infinite number of such solutions?

   (c) $Z_3(n) = 2 \cdot S(n)$? This equation up to 10^2 has 3 solutions: 24, 35, 85. There are an infinite number of such solutions?

3.11 General Residual Sequence

Let x, n two integer numbers. The *general residual* function is the product between $(x+C_1)(x+C_2)\cdots(x+C_{\varphi(n)})$, where C_k are the residual class of n which are relative primes to n. As we know, the relative prime factors of the number n are in number of $\varphi(n)$, where φ is Euler's totient function, [Weisstein, 2016b]. We can define the function $GR: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$,

$$GR(x,n) = \prod_{k=1}^{\varphi(n)} (x + C_k), \tag{3.15}$$

where the residual class C_k $(mod\ n)$ is the class C_k for which we have $(C_k, n) = 1$ (C_k relative prime to n), [Smarandache, 1992, 1995].

The fact that the residual class C_k is relative prime to n can be written also in the form $gcd(C_k, n) = 1$, i.e. the greatest common divisor of C_k and n is 1.

Program 3.14. General residual function

$$GR(x,n) := \left| \begin{array}{l} gr \leftarrow 1 \\ for\ k \in 1..n \\ \quad gr \leftarrow gr(x+k)\ \ if\ gcd(k,n){=}1 \\ return\ \ gr \end{array} \right.$$

Let $n = 2, 3, \ldots, 20$. General residual sequence for $x = 0$ is:

$$1, 2, 3, 24, 5, 720, 105, 2240, 189, 3628800, 385, 479001600, 19305, 896896,$$
$$2027025, 20922789888000, 85085, 6402373705728000, 8729721,$$

and for $x = 1$ is:

$$2, 6, 8, 120, 12, 5040, 384, 12960, 640, 39916800, 1152, 6227020800,$$
$$80640, 5443200, 10321920, 355687428096000, 290304,$$
$$121645100408832000, 38707200$$

and for $x = 2$ is:

$$3, 12, 15, 360, 21, 20160, 945, 45360, 1485, 239500800, 2457,$$
$$43589145600, 225225, 20217600, 34459425, 3201186852864000, 700245,$$
$$1216451004088320000, 115540425\ .$$

3.12 Goldbach–Smarandache Table

Goldbach's conjecture asserts that any even number > 4 is the sum of two primes, [Oliveira e Silva, 2016, Weisstein, 2016a].

Let be the sequence of numbers be:

$t_1 = 6$ the largest even number such that any other even number, not exceeding it, is the sum of two of the first 1 (one) odd prime 3; $6 = 3 + 3$;

$t_2 = 10$ the largest even number such that any other even number, not exceeding it, is the sum of two of the first 2 (two) odd primes 3, 5; $6 = 3+3$, $8 = 3 + 5$;

$t_3 = 14$ the largest even number such that any other even number, not exceeding it, is the sum of two of the first 3 (three) odd primes 3, 5, 7; $3 + 3 = 6$, $3 + 5 = 8$, $5 + 5 = 10$, $7 + 5 = 12$; $7 + 7 = 14$;

$t_4 = 18$ the largest even number such that any other even number, not exceeding it, is the sum of two of the first 4 (four) odd primes 3, 5, 7, 11; $6 = 3 + 3$, $8 = 3 + 5$, $10 = 3 + 7$, $12 = 5 + 7$, $14 = 7 + 7$, $16 = 5 + 11$, $18 = 11 + 7$;

$t_5 = \dots$.

Thus we have the sequence: 6, 10, 14, 18, 26, 30, 38, 42, 42, 54, 62, 74, 74, 90, ...

Table 3.38 contains the sum $prime_k + prime_j$, $k = 5, 6, \dots, 15$, $j = \overline{2,k}$, where *prime* is the vector of prime numbers obtained by the program 1.1 using the call $prime := SEPC(47)$. From this table one can see any even number n ($11 < n < 47$) to what amount of prime numbers equals.

+	11	13	17	19	23	29	31	37	41	43	47
3	14	16	20	22	26	32	34	40	44	46	50
5	16	18	22	24	28	34	36	42	46	48	52
7	18	20	24	26	30	36	38	44	48	50	54
11	22	24	28	30	34	40	42	48	52	54	58
13	0	26	30	32	36	42	44	50	54	56	60
17	0	0	34	36	40	46	48	54	58	60	64
19	0	0	0	38	42	48	50	56	60	62	66
23	0	0	0	0	46	52	54	60	64	66	70
29	0	0	0	0	0	58	60	66	70	72	76
31	0	0	0	0	0	0	62	68	72	74	78
37	0	0	0	0	0	0	0	74	78	80	84
41	0	0	0	0	0	0	0	0	82	84	88
43	0	0	0	0	0	0	0	0	0	86	90
47	0	0	0	0	0	0	0	0	0	0	94

Table 3.38: Goldbach–Smarandache table

Table 3.38 is generated by the program:

Program 3.15. Goldbach–Smarandache.

```
GSt(a,b) := | prime ← SEPC(b)
            | for k ∈ 2..last(prime)
            |   if prime_k ≥ a
            |     | ka ← k
            |     | break
            | for k ∈ ka..last(prime)
            |   for j ∈ 2..k
            |     g_{j-1,k-ka+1} ← prime_k + prime_j
            | return g
```

The call of the program *GSt* for obtaining the Table 3.38 is $GSt(11, 50)$.

We present a program that determines all possible combinations (apart from the addition's commutativity) of sums of two prime numbers that are equal to the given even number.

Program 3.16. search Goldbach–Smarandache table.

$$
SGSt(n) := \left|
\begin{array}{l}
return\ \text{"}Error\ n\ is\ odd\text{"}\ \ if\ \ \bmod(n,2) \neq 0\\
prime \leftarrow SEPC(n+2)\\
for\ k \in 2..\frac{n}{2}\\
\quad if\ prime_k \geq \frac{n}{2}\\
\quad \left| \begin{array}{l} ka \leftarrow k\\ break \end{array} \right.\\
h \leftarrow 1\\
for\ k \in last(prime)..ka\\
\quad for\ j \in 2..k\\
\quad\quad if\ n{=}prime_k + prime_j\\
\quad\quad \left| \begin{array}{l} g_{h,1} \leftarrow n\\ g_{h,2} \leftarrow \text{"="}\\ g_{h,3} \leftarrow prime_k\\ g_{h,4} \leftarrow \text{"+"}\\ g_{h,5} \leftarrow prime_j\\ h \leftarrow h+1\\ break \end{array} \right.\\
return\ \ g
\end{array}
\right.
$$

We present two of the program 3.16, calls, for numbers 556 and 346. The number 556 decompose in 11 sums of prime numbers, and 346 in 9 sums of prime numbers.

$$
SGSt(556) = \begin{pmatrix}
556 & \text{"="} & 509 & \text{"+"} & 47\\
556 & \text{"="} & 503 & \text{"+"} & 53\\
556 & \text{"="} & 467 & \text{"+"} & 89\\
556 & \text{"="} & 449 & \text{"+"} & 107\\
556 & \text{"="} & 443 & \text{"+"} & 113\\
556 & \text{"="} & 419 & \text{"+"} & 137\\
556 & \text{"="} & 389 & \text{"+"} & 167\\
556 & \text{"="} & 383 & \text{"+"} & 173\\
556 & \text{"="} & 359 & \text{"+"} & 197\\
556 & \text{"="} & 317 & \text{"+"} & 239\\
556 & \text{"="} & 293 & \text{"+"} & 263
\end{pmatrix}.
$$

$$SGSt(346)=\begin{pmatrix}346 & "=" & 317 & "+" & 29\\ 346 & "=" & 293 & "+" & 53\\ 346 & "=" & 263 & "+" & 83\\ 346 & "=" & 257 & "+" & 89\\ 346 & "=" & 239 & "+" & 107\\ 346 & "=" & 233 & "+" & 113\\ 346 & "=" & 197 & "+" & 149\\ 346 & "=" & 179 & "+" & 167\\ 346 & "=" & 173 & "+" & 173\end{pmatrix}.$$

The following program counts for any natural even number the number of possible unique decomposition (apart from the addition's commutativity) in the sum of two primes.

Program 3.17. for counting the decompositions of n (natural even number) in the sum of two primes.

```
NGSt(n) := | return "Err. n is odd" if  mod(n,2) = 1
           | prime ← SEPC(n)
           | h ← 0
           | for k ∈ last(prime)..2
           |   for j ∈ k..2
           |     if n=prime_k + prime_j
           |        | h ← h+1
           |        | break
           | return h
```

The call of this program using the commands:

$$n:=6,8..100\quad ngs_{\frac{n}{2}-2}:=NGSt(n)$$

will provide the Goldbach–Smarandache series:

$$ngs^{\mathrm{T}}=(1,1,2,1,2,2,2,2,3,3,3,2,3,2,4,4,2,3,4,3,4,5,4,\\ 3,5,3,4,6,3,5,6,2,5,6,5,5,7,4,5,8,5,4,9,4,5,7,3,6)\ .$$

3.13 Vinogradov–Smarandache Table

Vinogradov conjecture: Every sufficiently large odd number is a sum of three primes. Vinogradov proved in 1937 that any odd number greater than $3^{3^{15}}$ satisfies this conjecture.

Waring's prime number conjecture: Every odd integer n is a prime or the sum of three primes.

Let be the sequence of numbers:

v_1 = is the largest odd number such that any odd number ≥ 9, not exceeding it, is the sum of three of the first 1 (one) odd prime, i.e. the odd prime 3;

v_2 = is the largest odd number such that any odd number ≥ 9, not exceeding it, is the sum of three of the first 2 (two) odd prime, i.e. the odd primes 3, 5;

v_3 = is the largest odd number such that any odd number ≥ 9, not exceeding it, is the sum of three of the first 3 (three) odd prime, i.e. the odd primes 3, 5, 7;

v_4 = is the largest odd number such that any odd number ≥ 9, not exceeding it, is the sum of three of the first 4 (four) odd prime, i.e. the odd primes 3, 5, 7, 11;

$v_5 = \ldots$.

Thus we have the sequence: 9, 15, 21, 29, 39, 47, 57, 65, 71, 93, 99, 115, 129, 137, 143, 149, 183, 189, 205, 219, 225, 241, 251, 269, 287, 309, 317, 327, 335, 357, 371, 377, 417, 419, 441, 459, 465, 493, 503, 509, 543, 545, 567, 587, 597, 609, 621, 653, 657, 695, 701, 723, 725, 743, 749, 755, 785 ... , [Sloane, 2014, A007962].

The table gives you in how many different combinations an odd number is written as a sum of three odd primes, and in what combinations.

Program 3.18. that generates the Vinogradov–Smarandache Table with p prime fixed between the limits a and b.

```
VSt(p,a,b) := | prime ← SEPC(b)
              | for k ∈ 2..last(prime)
              |   if prime_k ≥ a
              |     | ka ← k
              |     | break
              | for k ∈ ka..last(prime)
              |   for j ∈ 2..k
              |     g_{j−1,k−ka+1} ← p + prime_k + prime_j
              | return g
```

With this program, we generate the Vinogradov–Smarandache tables for $p = 3, 5, 7, 11$ and $a = 13$, $b = 45$.

$$VSt(3,13,45)=\begin{pmatrix}19&23&25&29&35&37&43&47&49\\21&25&27&31&37&39&45&49&51\\23&27&29&33&39&41&47&51&53\\27&31&33&37&43&45&51&55&57\\29&33&35&39&45&47&53&57&59\\0&37&39&43&49&51&57&61&63\\0&0&41&45&51&53&59&63&65\\0&0&0&49&55&57&63&67&69\\0&0&0&0&61&63&69&73&75\\0&0&0&0&0&65&71&75&77\\0&0&0&0&0&0&77&81&83\\0&0&0&0&0&0&0&85&87\\0&0&0&0&0&0&0&0&89\end{pmatrix},$$

$$VSt(5,13,45)=\begin{pmatrix}21&25&27&31&37&39&45&49&51\\23&27&29&33&39&41&47&51&53\\25&29&31&35&41&43&49&53&55\\29&33&35&39&45&47&53&57&59\\31&35&37&41&47&49&55&59&61\\0&39&41&45&51&53&59&63&65\\0&0&43&47&53&55&61&65&67\\0&0&0&51&57&59&65&69&71\\0&0&0&0&63&65&71&75&77\\0&0&0&0&0&67&73&77&79\\0&0&0&0&0&0&79&83&85\\0&0&0&0&0&0&0&87&89\\0&0&0&0&0&0&0&0&91\end{pmatrix},$$

$$VSt(7,13,45)=\begin{pmatrix}23&27&29&33&39&41&47&51&53\\25&29&31&35&41&43&49&53&55\\27&31&33&37&43&45&51&55&57\\31&35&37&41&47&49&55&59&61\\33&37&39&43&49&51&57&61&63\\0&41&43&47&53&55&61&65&67\\0&0&45&49&55&57&63&67&69\\0&0&0&53&59&61&67&71&73\\0&0&0&0&65&67&73&77&79\\0&0&0&0&0&69&75&79&81\\0&0&0&0&0&0&81&85&87\\0&0&0&0&0&0&0&89&91\\0&0&0&0&0&0&0&0&93\end{pmatrix},$$

$$VSt(11,13,45)=\begin{pmatrix} 27 & 31 & 33 & 37 & 43 & 45 & 51 & 55 & 57 \\ 29 & 33 & 35 & 39 & 45 & 47 & 53 & 57 & 59 \\ 31 & 35 & 37 & 41 & 47 & 49 & 55 & 59 & 61 \\ 35 & 39 & 41 & 45 & 51 & 53 & 59 & 63 & 65 \\ 37 & 41 & 43 & 47 & 53 & 55 & 61 & 65 & 67 \\ 0 & 45 & 47 & 51 & 57 & 59 & 65 & 69 & 71 \\ 0 & 0 & 49 & 53 & 59 & 61 & 67 & 71 & 73 \\ 0 & 0 & 0 & 57 & 63 & 65 & 71 & 75 & 77 \\ 0 & 0 & 0 & 0 & 69 & 71 & 77 & 81 & 83 \\ 0 & 0 & 0 & 0 & 0 & 73 & 79 & 83 & 85 \\ 0 & 0 & 0 & 0 & 0 & 0 & 85 & 89 & 91 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 93 & 95 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 97 \end{pmatrix}.$$

We present a program that determines all possible combinations of Vinogradov–Smarandache Tables, so that the odd number to be written as the sum of 3 primes. Because the Vinogradov–Smarandache Tables are tridimensional, we fix the first factor in the triplet of prime numbers so that the program will determine the other two prime numbers, so that the sum of 3 primes to be n.

Program 3.19. search Vinogradov–Smarandache table.

```
SVSt(p,n) := | return "Error n is odd" if mod(n,2)=0
             | prime ← SEPC(n − p)
             | for k ∈ 2..(n+1)/2
             |   if prime_k ≥ (n+1)/2
             |     | ka ← k
             |     | break
             | h ← 1
             | for k ∈ last(prime)..ka
             |   for j ∈ 2..k
             |     if n=p + prime_k + prime_j
             |       | g_{h,1} ← n
             |       | g_{h,2} ← " = "
             |       | g_{h,3} ← p
             |       | g_{h,4} ← " + "
             |       | g_{h,5} ← prime_k
             |       | g_{h,6} ← " + "
             |       | g_{h,7} ← prime_j
             |       | h ← h + 1
             |       | break
             | return g
```

For illustration we determine all possible cases of three prime numbers that add up to be 559..

$$SVSt(3,559)=\begin{pmatrix} 559 & "=" & 3 & "+" & 509 & "+" & 47\\ 559 & "=" & 3 & "+" & 503 & "+" & 53\\ 559 & "=" & 3 & "+" & 467 & "+" & 89\\ 559 & "=" & 3 & "+" & 449 & "+" & 107\\ 559 & "=" & 3 & "+" & 443 & "+" & 113\\ 559 & "=" & 3 & "+" & 419 & "+" & 137\\ 559 & "=" & 3 & "+" & 389 & "+" & 167\\ 559 & "=" & 3 & "+" & 383 & "+" & 173\\ 559 & "=" & 3 & "+" & 359 & "+" & 197\\ 559 & "=" & 3 & "+" & 317 & "+" & 239\\ 559 & "=" & 3 & "+" & 293 & "+" & 263 \end{pmatrix},$$

$$SVSt(5,559)=\begin{pmatrix} 559 & "=" & 5 & "+" & 547 & "+" & 7\\ 559 & "=" & 5 & "+" & 541 & "+" & 13\\ 559 & "=" & 5 & "+" & 523 & "+" & 31\\ 559 & "=" & 5 & "+" & 487 & "+" & 67\\ 559 & "=" & 5 & "+" & 457 & "+" & 97\\ 559 & "=" & 5 & "+" & 397 & "+" & 157\\ 559 & "=" & 5 & "+" & 373 & "+" & 181\\ 559 & "=" & 5 & "+" & 331 & "+" & 223\\ 559 & "=" & 5 & "+" & 313 & "+" & 241\\ 559 & "=" & 5 & "+" & 283 & "+" & 271\\ 559 & "=" & 5 & "+" & 277 & "+" & 277 \end{pmatrix},$$

$$\vdots$$

$$SVSt(181,559)=\begin{pmatrix} 559, & "=" & 181 & "+" & 197 & "+" & 181 \end{pmatrix}.$$

$$SVSt(191,559)=0\,.$$

Program 3.20. for counting the decomposition of n (odd natural numbers $n \geq 3$) in the sum of three primes.

```
NVSt(n) := | return "Error n is even" if  mod (n,2) = 0
           | prime ← SEPC(n+1)
           | h ← 0
           | for k ∈ last(prime)..2
           |   for j ∈ k..2
           |     for i ∈ j..2
```

$$\begin{array}{|l}
\quad\quad if\ \ n{=}prime_k + prime_j + prime_i \\
\quad\quad \left| \begin{array}{l} h \leftarrow h+1 \\ break \end{array} \right. \\
return\ \ h
\end{array}$$

The call of this program using the controls

$$n := 3, 5..100 \quad nvs_{\frac{n-1}{2}} := NVSt(n)$$

will provide the Vinogradov–Smarandache series

$$nvs^{\mathrm{T}} = (0,0,0,1,1,2,2,3,3,4,4,5,6,7,6,8,7,9,10,10,10,11,12,12,\\
14,16,14,16,16,16,18,20,20,20,21,21,21,27,24,25,28,\\
27,28,33,29,32,35,34,30)\ .$$

3.14 Smarandacheian Complements

Let $g : A \to A$ be a strictly increasing function, and let "$\sim$" be a given internal law on A. Then $f : A \to A$ is a smarandacheian complement with respect to the function g and the internal law "$\sim$" if: $f(x)$ is the smallest k such that there exists a z in A so that $x \sim k = g(z)$.

3.14.1 Square Complements

Definition 3.21. For each integer n to find the smallest integer k such that $k \cdot n$ is a perfect square.

Observation 3.22. All these numbers are square free.

Numbers square complements in between 10 and 10^2 are: 10, 11, 3, 13, 14, 15, 1, 17, 2, 19, 5, 21, 22, 23, 6, 1, 26, 3, 7, 29, 30, 31, 2, 33, 34, 35, 1, 37, 38, 39, 10, 41, 42, 43, 11, 5, 46, 47, 3, 1, 2, 51, 13, 53, 6, 55, 14, 57, 58, 59, 15, 61, 62, 7, 1, 65, 66, 67, 17, 69, 70, 71, 2, 73, 74, 3, 19, 77, 78, 79, 5, 1, 82, 83, 21, 85, 86, 87, 22, 89, 10, 91, 23, 93, 94, 95, 6, 97, 2, 11, 1 . These were generated with the program 3.27, using the command: $mC(2,10,100)^{\mathrm{T}} =$.

3.14.2 Cubic Complements

Definition 3.23. For each integer n to find the smallest integer k such that $k \cdot n$ is a perfect cub.

Observation 3.24. All these numbers are cube free.

Numbers cubic complements in between 1 and 40 are: 1, 2, 3, 2, 5, 6, 7, 1, 3, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 5, 26, 1, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40 . These were generated with the program 3.27, using the command: $mC(3,1,40)^{\mathrm{T}} =$.

3.14.3 $m-$power Complements

Definition 3.25. For each integer n to find the smallest integer k such that $k\cdot n$ is a $m-$power.

Observation 3.26. All these numbers are $m-$power free.

Program 3.27. for generating the numbers $m-$power complements.

$$
mC(m,n_a,n_b):=\left|\begin{array}{l}
for\ n\in n_a..n_b\\
\quad\left|\begin{array}{l}
for\ k\in 1..n\\
\quad\left|\begin{array}{l}
kn\leftarrow k\cdot n\\
break\ \ if\ trunc\left(\sqrt[m]{kn}\right)^m = kn
\end{array}\right.\\
mc_{n-n_a+1}\leftarrow k
\end{array}\right.\\
return\ mc
\end{array}\right.
$$

which uses Mathcad function *trunc*.

Numbers 5−power complements in between 25 and 65 are: 25, 26, 9, 28, 29, 30, 31, 1, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 16, 65 . These were generated with the program 3.27, using the command: $mC(5,25,65)^{\mathrm{T}} =$.

3.15 $m-$factorial Complements

Definition 3.28. For each $n\in\mathbb{N}^*$ to find the smallest k such that $k\cdot n$ is a $m-$factorial, where $m\in\mathbb{N}^*$.

Program 3.29. for generating the series $m-$factorial complements.

$$
mfC(m,n_a,n_b):=\left|\begin{array}{l}
for\ n\in n_a..n_b\\
\quad for\ j\in 1..n\\
\quad\quad\left|\begin{array}{l}
k\leftarrow \dfrac{kf(j,m)}{n}\\
if\ k=trunc(k)\\
\quad\left|\begin{array}{l}
mfc_{n-n_a+1}\leftarrow k\\
break
\end{array}\right.
\end{array}\right.\\
return\ \ mfc
\end{array}\right.
$$

which uses the program 2.15 and Mathcad function *trunc*.

Example 3.30. The series of *factorial complements* numbers from 1 to 25 is obtained with the command $mfc(1,1,25)$: 1, 1, 2, 6, 24, 1, 720, 3, 80, 12, 3628800, 2, 479001600, 360, 8, 45, 20922789888000, 40, 6402373705728000, 6, 240, 1814400, 5288917409079652, 1, 145152 .

Example 3.31. The series of *double factorial complements* numbers from 1 to 30 is obtained with the command $mfc(2,1,30)$: 1, 1, 1, 2, 3, 8, 15, 1, 105, 384, 945, 4, 10395, 46080, 1, 3, 2027025, 2560, 34459425, 192, 5, 3715891200, 13749310575, 2, 81081, 1961990553600, 35, 23040, 213458046676875, 128 .

Example 3.32. The series of *triple factorial complements* numbers from 1 to 35 is obtained with the command $mfc(3,1,35)$: 1, 1, 1, 1, 2, 3, 4, 10, 2, 1, 80, 162, 280, 2, 1944, 5, 12320, 1, 58240, 4, 524880, 40, 4188800, 81, 167552, 140, 6, 1, 2504902400, 972, 17041024000, 385, 214277011200, 6160, 8 .

3.16 Prime Additive Complements

Definition 3.33. For each $n \in \mathbb{N}^*$ to find the smallest k such that $n+k \in \mathbb{P}_{\geq 2}$.

Program 3.34. for generating the series of *prime additive complements* numbers.

$$paC(n_a,n_b) := \left| \begin{array}{l} for\ n \in n_a..n_b \\ \quad pac_{n-n_a+1} \leftarrow spp(n) - n \\ return\ \ pac \end{array} \right.$$

where spp is the program 2.42.

Example 3.35. The series of *prime additive complements* numbers between limits $n_a = 1$ and $n_b = 53$ are generated with the command $paC(1,53)$ = are: 1, 0, 0, 1, 0, 1, 0, 3, 2, 1, 0, 1, 0, 3, 2, 1, 0, 1, 0, 3, 2, 1, 0, 5, 4, 3, 2, 1, 0, 1, 0, 5, 4, 3, 2, 1, 0, 3, 2, 1, 0, 1, 0, 3, 2, 1, 0, 5, 4, 3, 2, 1, 0 .

Example 3.36. The series of *prime additive complements* numbers between limits $n_a = 114$ and $n_b = 150$ are generated with the command $paC(114,150)$ = are: 13, 12, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1, 0, 3, 2, 1, 0, 5, 4, 3, 2, 1, 0, 1, 0, 9, 8, 7, 6, 5, 4, 3, 2, 1, 0, 1 .

3.17 Sequence of Position

Let $a = \overline{a_m a_{m-1} \ldots a_0}_{(10)}$ be a decimal integer, and $0 \le k \le 9$ a digit. The position is defined as follows:

$$u(k,"min",a) = \begin{cases} \min\limits_{i=\overline{0,m}} \{i \mid a_i = k\}, & \text{if } \exists\, i \text{ such that } a_i = k; \\ -1, & \text{if } \nexists\, i \text{ such that } a_i = k. \end{cases} \tag{3.16}$$

$$u(k,"max",a) = \begin{cases} \max\limits_{i=\overline{0,m}} \{i \mid a_i = k\}, & \text{if } \exists\, i \text{ such that } a_i = k; \\ -1, & \text{if } \nexists\, i \text{ such that } a_i = k. \end{cases} \tag{3.17}$$

Let $x = \{x_1, x_2, \ldots, x_n\}$ be the sequence of positive integer numbers, then the sequence of position is defined as follows:

$$U(k,"min",x) = \{u(k,"min",x_1), u(k,"min",x_2), \ldots, u(k,"min",x_n)\}$$

or

$$U(k,"max",x) = \{u(k,"max",x_1), u(k,"max",x_2), \ldots, u(k,"max",x_n)\}\ .$$

Program 3.37. for functions (3.16) and (3.17).

```
u(k, minmax, a) := | d ← dn(a, 10)
                   | m ← last(d)
                   | j ← 0
                   | for i ∈ 1..m
                   |   if d_i=k
                   |     | j ← j + 1
                   |     | u_j ← m − i
                   | if j > 0
                   |   | return min(u) if minmax="min"
                   |   | return max(u) if minmax="max"
                   | return −1 otherwise
```

This program uses the program *dn*, 2.2.

Program 3.38. generate sequence of position.

```
U(k, minmax, x) := | return "Err." if k < 0 ∨ k > 9
                   | for j ∈ 1..last(x)
                   |   | U_j ← u(k, minmax, x_j)
                   | return U
```

Examples of series of position:

1. Random sequence of numbers with less than 5 digits: $n := 10$ $j := 1..n$ $x_j := floor(rnd(10^{10}))$, where *rnd*($a$) is the Mathcad function for generating random numbers with uniform distribution from 0 to a, then x^{T} =(2749840272, 8389146924, 3712396081, 2325329044, 2316651791, 9710168987, 229116575, 1518263844, 92574637, 6438703378). In this situation we obtain:

   $U(7, "min", x)^{\mathrm{T}}$ =(1, −1, 8, −1, 2, 0, 1, −1, 0, 1);

   $U(7, "max", x)^{\mathrm{T}}$ =(8, −1, 8, −1, 2, 8, 1, −1, 4, 5).

2. Sequence of prime numbers $p := submatrix(prime, 1, 33, 1, 1)$ i.e. p^{T} =(2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113, 127, 131, 137), then:

   $U(1, "min", p)^{\mathrm{T}}$ =(−1, −1, −1, −1, 0, 1, 1, 1, −1, −1, 0, −1, 0, −1, −1, −1, −1, 0, −1, 0, −1, −1, −1, −1, −1, 0, 2, 2, 2, 1, 2, 0, 2);

   $U(1, "max", p)^{\mathrm{T}}$ =(−1, −1, −1, −1, 1, 1, 1, 1, −1, −1, 0, −1, 0, −1, −1, −1, −1, 0, −1, 0, −1, −1, −1, −1, −1, 2, 2, 2, 2, 2, 2, 2, 2).

3. Sequence of factorials $n := 16$ $j := 1..n$ $f_j := j!$ i.e. f^{T} =(1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800, 39916800, 479001600, 6227020800, 87178291200, 1307674368000, 20922789888000), then:

   $U(2, "min", f)^{\mathrm{T}}$ =(−1, 0, −1, 1, 1, 1, −1, 1, 3, 4, −1, −1, 4, 2, −1, 9);

   $U(2, "max", f)^{\mathrm{T}}$ =(−1, 0, −1, 1, 1, 1, −1, 1, 3, 4, −1, −1, 8, 5, −1, 13).

4. Sequence of *Left Mersenne* numbers $n := 22$ $j := 1..n$ $M\ell_j := 2^j - 1$ i.e. $M\ell^{\mathrm{T}}$ =(1, 3, 7, 15, 31, 63, 127, 255, 511, 1023, 2047, 4095, 8191, 16383, 32767, 65535, 131071, 262143, 524287, 1048575, 2097151, 4194303), then:

   $U(1, "min", M\ell)^{\mathrm{T}}$ =(0, −1, −1, 1, 0, −1, 2, −1, 0, 3, −1, −1, 0, 4, −1, −1, 0, 2, −1, 6, 0, 5);

   $U(1, "max", M\ell)^{\mathrm{T}}$ =(0, −1, −1, 1, 0, −1, 2, −1, 1, 3, −1, −1, 2, 4, −1, −1, 5, 2, −1, 6, 2, 5).

5. Sequence of *Right Mersenne* numbers $n := 22$ $j := 1..n$ $Mr_j := 2^j + 1$ i.e. Mr^{T} =(3, 5, 9, 17, 33, 65, 129, 257, 513, 1025, 2049, 4097, 8193, 16385, 32769, 65537, 131073, 262145, 524289, 1048577, 2097153, 4194305), then:

   $U(1, "min", Mr)^{\mathrm{T}}$ =(−1, −1, −1, 1, −1, −1, 2, −1, 1, 3, −1, −1, 2, 4, −1, −1, 3, 2, −1, 6, 2, 5);

$U(1,"max",Mr)^{\mathrm{T}} =$(–1, –1, –1, 1, –1, –1, 2, –1, 1, 3, –1, –1, 2, 4, –1, –1, 5, 2, –1, 6, 2, 5).

6. Sequence Fibonacci numbers $n := 24$ $j := 1..n$ $F_1 := 1$ $F_2 := 1$ $F_{j+2} := F_{j+1} + F_j$ i.e. $F^{\mathrm{T}} =$(1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, 6765, 10946, 17711, 28657, 46368, 75025, 121393), then:

   $U(1,"min",F)^{\mathrm{T}} =$(0, 0, –1, –1, –1, –1, 1, 0, –1, –1, –1, 2, –1, –1, 1, –1, 3, –1, 0, –1, 4, 0, –1, –1, –1, 3);

   $U(1,"max",F)^{\mathrm{T}} =$(0, 0, –1, –1, –1, –1, 1, 0, –1, –1, –1, 2, –1, –1, 1, –1, 3, –1, 2, –1, 4, 4, –1, –1, –1, 5).

7. Sequence of numbers type n^n: $n := 13$ $j := 1..n$ $N_j := j^j$ i.e. $N^{\mathrm{T}} =$(1, 4, 27, 256, 3125, 46656, 823543, 16777216, 387420489, 10000000000, 285311670611, 8916100448256, 302875106592253), then:

   $U(6,"min",N)^{\mathrm{T}} =$(–1, –1, –1, 0, –1, 0, –1, 0, –1, –1, 2, 0, 6);

   $U(6,"max",N)^{\mathrm{T}} =$(–1, –1, –1, 0, –1, 3, –1, 6, –1, –1, 5, 9, 6);

8. Sequence of primorial numbers $n := 33$ $j := 1..n$ $P_j := kP(p_j, 1)$, where we used the program *kP*, 1.14, then:

   $U(6,"min",P)^{\mathrm{T}} =$(–1, 0, –1, –1, –1, –1, –1, 2, –1, 5, 7, –1, 7, 5, 17, –1, 16, 9, 10, 14, 23, 17, 23, 19, 28, 31, –1, 36, 36, 32, 42, 38, 44);

   $U(6,"max",N)^{\mathrm{T}} =$(–1, 0, –1, –1, –1, –1, –1, 5, –1, 9, 7, –1, 7, 10, 17, –1, 16, 9, 10, 15, 23, 26, 31, 31, 31, 34, –1, 40, 36, 44, 42, 45, 44).

Study:

1. $\{U(k,"min",x)\}$, where $\{x\}_n$ is the sequence of numbers (double factorial, triple factorial, ..., Fibonacci, Tribonacci, Tetranacci, ..., primorial, double primorial, triple primorial, ..., etc). Convergence, monotony.

2. $\{U(k,"max",x)\}$, where $\{x\}_n$ is the sequence of numbers (double factorial, triple factorial, ..., Fibonacci, Tribonacci, Tetranacci, ..., primorial, double primorial, triple primorial, ..., etc). Convergence, monotony.

3.18 The General Periodic Sequence

Let $\mathscr{S}$ be a finite set, and $f : \mathscr{S} \to \mathscr{S}$ be a function defined for all elements of $\mathscr{S}$. There will always be a periodic sequence whenever we repeat the composition of the function f with itself more times than $card(\mathscr{S})$, accordingly to

the box principle of Dirichlet. The invariant sequence is considered a periodic sequence whose period length has one term.

Thus the General Periodic Sequence is defined as:

- $a_1 = f(s)$, where $s \in \mathscr{S}$;
- $a_2 = f(a_1) = f(f(s))$, where $s \in \mathscr{S}$;
- $a_3 = f(a_2) = f(f(a_1)) = f(f(f(s)))$, where $s \in \mathscr{S}$;
- and so on.

We particularize $\mathscr{S}$ and f to study interesting cases of this type of sequences, [Popov, 1996/7].

3.18.1 Periodic Sequences

The n–Digit Periodic Sequence

Let n_1 be an integer of at most two digits and let n_1' be its digital reverse. One defines the absolute value $n_2 = \left|n_1 - n_1'\right|$. And so on: $n_3 = \left|n_2 - n_2'\right|$, etc. If a number n has one digit only, one considers its reverse as $n \times 10$ (for example: 5, which is 05, reversed will be 50). This sequence is periodic. Except the case when the two digits are equal, and the sequence becomes: $n_1, 0, 0, 0, \ldots$ the iteration always produces a loop of length 5, which starts on the second or the third term of the sequence, and the period is 9, 81, 63, 27, 45, or a cyclic permutation thereof.

Function 3.39. for periodic sequence.

$$PS(n) := |n - Reverse(n)| \ ,$$

where it uses the function *Reverse*, 1.6.

Program 3.40. of the application of function *PS*, 3.39 of r times the elements vector v.

```
PPS(v,r) := | for k ∈ 1..last(v)
            |   | a_{k,1} ← v_k
            |   | for j = 1..r − 1
            |   |   | sw ← 0
            |   |   | for i = 1..j − 1 if j ≥ 2
            |   |   |    if a_{k,i}=a_{k,j}
            |   |   |      | a_{k,j} ← 0
            |   |   |      | break
```

$$
\left|\begin{array}{l}
\left|\begin{array}{l}
\left|\begin{array}{l}
if\ a_{k,j} \neq 0 \\
\left|\begin{array}{l} a_{k,j+1} \leftarrow PS(a_{k,j}) \\ sw \leftarrow sw+1 \end{array}\right. \\
break\ \ if\ sw{=}0
\end{array}\right.
\end{array}\right. \\
return\ \ a
\end{array}\right.
$$

The following table has resulted from commands: $k := 10..99$ $v_{k-9} := k$ $PPS(v,7) =$. In table through $PS(n)^k$ understand $PS(PS(\ldots PS(n)))$ of k times.

Table 3.39: The two–digit periodic sequence

n	$PS(n)$	$PS^2(n)$	$PS^3(n)$	$PS^4(n)$	$PS^5(n)$	$PS^6(n)$
10	9	0	0	0	0	0
11	0	0	0	0	0	0
12	9	0	0	0	0	0
13	18	63	27	45	9	0
14	27	45	9	0	0	0
15	36	27	45	9	0	0
16	45	9	0	0	0	0
17	54	9	0	0	0	0
18	63	27	45	9	0	0
19	72	45	9	0	0	0
20	18	63	27	45	9	0
21	9	0	0	0	0	0
22	0	0	0	0	0	0
23	9	0	0	0	0	0
24	18	63	27	45	9	0
25	27	45	9	0	0	0
26	36	27	45	9	0	0
27	45	9	0	0	0	0
28	54	9	0	0	0	0
29	63	27	45	9	0	0
30	27	45	9	0	0	0
31	18	63	27	45	9	0
32	9	0	0	0	0	0
33	0	0	0	0	0	0
34	9	0	0	0	0	0
35	18	63	27	45	9	0
36	27	45	9	0	0	0

Continued on next page

n	$PS(n)$	$PS^2(n)$	$PS^3(n)$	$PS^4(n)$	$PS^5(n)$	$PS^6(n)$
37	36	27	45	9	0	0
38	45	9	0	0	0	0
39	54	9	0	0	0	0
40	36	27	45	9	0	0
41	27	45	9	0	0	0
42	18	63	27	45	9	0
43	9	0	0	0	0	0
44	0	0	0	0	0	0
45	9	0	0	0	0	0
46	18	63	27	45	9	0
47	27	45	9	0	0	0
48	36	27	45	9	0	0
49	45	9	0	0	0	0
50	45	9	0	0	0	0
51	36	27	45	9	0	0
52	27	45	9	0	0	0
53	18	63	27	45	9	0
54	9	0	0	0	0	0
55	0	0	0	0	0	0
56	9	0	0	0	0	0
57	18	63	27	45	9	0
58	27	45	9	0	0	0
59	36	27	45	9	0	0
60	54	9	0	0	0	0
61	45	9	0	0	0	0
62	36	27	45	9	0	0
63	27	45	9	0	0	0
64	18	63	27	45	9	0
65	9	0	0	0	0	0
66	0	0	0	0	0	0
67	9	0	0	0	0	0
68	18	63	27	45	9	0
69	27	45	9	0	0	0
70	63	27	45	9	0	0
71	54	9	0	0	0	0
72	45	9	0	0	0	0
73	36	27	45	9	0	0
74	27	45	9	0	0	0
75	18	63	27	45	9	0

Continued on next page

n	$PS(n)$	$PS^2(n)$	$PS^3(n)$	$PS^4(n)$	$PS^5(n)$	$PS^6(n)$
76	9	0	0	0	0	0
77	0	0	0	0	0	0
78	9	0	0	0	0	0
79	18	63	27	45	9	0
80	72	45	9	0	0	0
81	63	27	45	9	0	0
82	54	9	0	0	0	0
83	45	9	0	0	0	0
84	36	27	45	9	0	0
85	27	45	9	0	0	0
86	18	63	27	45	9	0
87	9	0	0	0	0	0
88	0	0	0	0	0	0
89	9	0	0	0	0	0
90	81	63	27	45	9	0
91	72	45	9	0	0	0
92	63	27	45	9	0	0
93	54	9	0	0	0	0
94	45	9	0	0	0	0
95	36	27	45	9	0	0
96	27	45	9	0	0	0
97	18	63	27	45	9	0
98	9	0	0	0	0	0
99	0	0	0	0	0	0

1. The 3–digit periodic sequence (domain $10^2 \le n_1 \le 10^3 - 1$):
   - there are 90 symmetric integers, 101, 111, 121, ..., for which $n_2 = 0$;
   - all other initial integers iterate into various entry points of the same periodic subsequence (or a cyclic permutation thereof) of five terms: 99, 891, 693, 297, 495.

   For example we take 3–digits prime numbers and we study periodic sequence. The following table has resulted from commands: $p := submatrix(prime, 26, 168, 1, 1)$ $PPS(p, 7) =$. In table through $PS(n)^k$ understand $PS(PS(\ldots PS(n)))$ of k times.

Table 3.40: Primes with 3–digits periodic sequences

n	$PS(n)$	$PS^2(n)$	$PS^3(n)$	$PS^4(n)$	$PS^5(n)$	$PS^6(n)$
101	0	0	0	0	0	0
103	198	693	297	495	99	0
107	594	99	0	0	0	0
109	792	495	99	0	0	0
113	198	693	297	495	99	0
127	594	99	0	0	0	0
131	0	0	0	0	0	0
137	594	99	0	0	0	0
139	792	495	99	0	0	0
149	792	495	99	0	0	0
151	0	0	0	0	0	0
157	594	99	0	0	0	0
163	198	693	297	495	99	0
167	594	99	0	0	0	0
173	198	693	297	495	99	0
179	792	495	99	0	0	0
181	0	0	0	0	0	0
191	0	0	0	0	0	0
193	198	693	297	495	99	0
197	594	99	0	0	0	0
199	792	495	99	0	0	0
211	99	0	0	0	0	0
223	99	0	0	0	0	0
227	495	99	0	0	0	0
229	693	297	495	99	0	0
233	99	0	0	0	0	0
239	693	297	495	99	0	0
241	99	0	0	0	0	0
251	99	0	0	0	0	0
257	495	99	0	0	0	0
263	99	0	0	0	0	0
269	693	297	495	99	0	0
271	99	0	0	0	0	0
277	495	99	0	0	0	0
281	99	0	0	0	0	0
283	99	0	0	0	0	0
293	99	0	0	0	0	0

Continued on next page

n	$PS(n)$	$PS^2(n)$	$PS^3(n)$	$PS^4(n)$	$PS^5(n)$	$PS^6(n)$
307	396	297	495	99	0	0
311	198	693	297	495	99	0
313	0	0	0	0	0	0
317	396	297	495	99	0	0
331	198	693	297	495	99	0
337	396	297	495	99	0	0
347	396	297	495	99	0	0
349	594	99	0	0	0	0
353	0	0	0	0	0	0
359	594	99	0	0	0	0
367	396	297	495	99	0	0
373	0	0	0	0	0	0
379	594	99	0	0	0	0
383	0	0	0	0	0	0
389	594	99	0	0	0	0
397	396	297	495	99	0	0
401	297	495	99	0	0	0
409	495	99	0	0	0	0
419	495	99	0	0	0	0
421	297	495	99	0	0	0
431	297	495	99	0	0	0
433	99	0	0	0	0	0
439	495	99	0	0	0	0
443	99	0	0	0	0	0
449	495	99	0	0	0	0
457	297	495	99	0	0	0
461	297	495	99	0	0	0
463	99	0	0	0	0	0
467	297	495	99	0	0	0
479	495	99	0	0	0	0
487	297	495	99	0	0	0
491	297	495	99	0	0	0
499	495	99	0	0	0	0
503	198	693	297	495	99	0
509	396	297	495	99	0	0
521	396	297	495	99	0	0
523	198	693	297	495	99	0
541	396	297	495	99	0	0
547	198	693	297	495	99	0

Continued on next page

n	$PS(n)$	$PS^2(n)$	$PS^3(n)$	$PS^4(n)$	$PS^5(n)$	$PS^6(n)$
557	198	693	297	495	99	0
563	198	693	297	495	99	0
569	396	297	495	99	0	0
571	396	297	495	99	0	0
577	198	693	297	495	99	0
587	198	693	297	495	99	0
593	198	693	297	495	99	0
599	396	297	495	99	0	0
601	495	99	0	0	0	0
607	99	0	0	0	0	0
613	297	495	99	0	0	0
617	99	0	0	0	0	0
619	297	495	99	0	0	0
631	495	99	0	0	0	0
641	495	99	0	0	0	0
643	297	495	99	0	0	0
647	99	0	0	0	0	0
653	297	495	99	0	0	0
659	297	495	99	0	0	0
661	495	99	0	0	0	0
673	297	495	99	0	0	0
677	99	0	0	0	0	0
683	297	495	99	0	0	0
691	495	99	0	0	0	0
701	594	99	0	0	0	0
709	198	693	297	495	99	0
719	198	693	297	495	99	0
727	0	0	0	0	0	0
733	396	297	495	99	0	0
739	198	693	297	495	99	0
743	396	297	495	99	0	0
751	594	99	0	0	0	0
757	0	0	0	0	0	0
761	594	99	0	0	0	0
769	198	693	297	495	99	0
773	396	297	495	99	0	0
787	0	0	0	0	0	0
797	0	0	0	0	0	0
809	99	0	0	0	0	0

Continued on next page

n	$PS(n)$	$PS^2(n)$	$PS^3(n)$	$PS^4(n)$	$PS^5(n)$	$PS^6(n)$
811	693	297	495	99	0	0
821	693	297	495	99	0	0
823	495	99	0	0	0	0
827	99	0	0	0	0	0
829	99	0	0	0	0	0
839	99	0	0	0	0	0
853	495	99	0	0	0	0
857	99	0	0	0	0	0
859	99	0	0	0	0	0
863	495	99	0	0	0	0
877	99	0	0	0	0	0
881	693	297	495	99	0	0
883	495	99	0	0	0	0
887	99	0	0	0	0	0
907	198	693	297	495	99	0
911	792	495	99	0	0	0
919	0	0	0	0	0	0
929	0	0	0	0	0	0
937	198	693	297	495	99	0
941	792	495	99	0	0	0
947	198	693	297	495	99	0
953	594	99	0	0	0	0
967	198	693	297	495	99	0
971	792	495	99	0	0	0
977	198	693	297	495	99	0
983	594	99	0	0	0	0
991	792	495	99	0	0	0
997	198	693	297	495	99	0

2. The 4–digit periodic sequence (domain $10^3 \leq n_1 \leq 10^4-1$), [Ibstedt, 1997]:

   - the largest number of iterations carried out in order to reach the first member of the loop is 18, and it happens for $n_1 = 1019$
   - iterations of 8818 integers result in one of the following loops (or a cyclic permutation thereof): 2178, 6534; or 90, 810, 630, 270, 450; or 909, 8181, 6363, 2727, 4545; or 999, 8991, 6993, 2997, 4995;
   - the other iterations ended up in the invariant 0.

3. The 5–digit periodic sequence (domain $10^4 \leq n_1 \leq 10^5 - 1$):
   - there are 920 integers iterating into the invariant 0 due to symmetries;
   - the other ones iterate into one of the following loops (or a cyclic permutation of these): 21978, 65934; or 990, 8910, 6930, 2970, 4950; or 9009, 81081, 63063, 27027, 45045; or 9999, 89991, 69993, 29997, 49995.
4. The 6–digit periodic sequence (domain $10^5 \leq n_1 \leq 10^6 - 1$):
   - there are 13667 integers iterating into the invariant 0 due to symmetries;
   - the longest sequence of iterations before arriving at the first loop is 53 for $n_1 = 100720$;
   - the loops have 2, 5, 9, or 18 terms.

3.18.2 The Subtraction Periodic Sequences

Let c be a positive integer. Start with a positive integer n, and let n' be its digital reverse. Put $n_1 = \left|n' - c\right|$, and let n_1' be its digital reverse. Put $n_2 = \left|n_1' - c\right|$, and let n_2' be its digital reverse. And so on. We shall eventually obtain a repetition.

For example, with $c = 1$ and $n = 52$ we obtain the sequence: 52, 24, 41, 13, 30, 02, 19, 90, 08, 79, 96, 68, 85, 57, 74, 46, 63, 35, 52, ... Here a repetition occurs after 18 steps, and the length of the repeating cycle is 18.

First example: $c = 1$, $10 \leq n \leq 999$. Every other member of this interval is an entry point into one of five cyclic periodic sequences (four of these are of length 18, and one of length 9). When n is of the form $11k$ or $11k-1$, then the iteration process results in 0.

Second example: $1 \leq c \leq 9$, $100 \leq n \leq 999$. For $c = 1, 2$, or 5 all iterations result in the invariant 0 after, sometimes, a large number of iterations. For the other values of c there are only eight different possible values for the length of the loops, namely 11, 22, 33, 50, 100, 167, 189, 200.

For $c = 7$ and $n = 109$ we have an example of the longest loop obtained: it has 200 elements, and the loop is closed after 286 iterations, [Ibstedt, 1997].

Program 3.41. for the subtraction periodic sequences.

$$SPS(n,c) := \left| \begin{array}{l} return\ \ |Reverse(n)\cdot 10 - c|\ \ if\ n < 10 \\ return\ \ |Reverse(n) - c|\ \ otherwise \end{array} \right.$$

The program use the function *Reverse*, 1.6.

Program 3.42. of the application of function f, 3.41 or 3.43 of r times the elements vector v with constant c.

```
PPS(v,c,r,f) := | for k ∈ 1..last(v)
                |   | a_{k,1} ← v_k
                |   | for j = 1..r−1
                |   |   | sw ← 0
                |   |   | for i = 1..j−1  if j ≥ 2
                |   |   |   if a_{k,i}=a_{k,j}
                |   |   |     | a_{k,j} ← 0
                |   |   |     | break
                |   |   | if a_{k,j} ≠ 0
                |   |   |   | a_{k,j+1} ← f(a_{k,j}, c)
                |   |   |   | sw ← sw+1
                |   |   | if sw=0
                |   |   |   | a_{k,j} ← a_{k,i}  if a_{k,j}=0
                |   |   |   | break
                | return a
```

With the commands $k := 10..35$ $v_{k-9} := k$ $PPS(v, 6, 10, SPS) =$, where *PPS*, 3.42, one obtains the table:

Table 3.41: 2–digits substraction periodic sequences

n	$f(n)$	$f^2(n)$	$f^3(n)$	$f^4(n)$	$f^5(n)$	$f^6(n)$	$f^7(n)$	$f^8(n)$	$f^9(n)$	$f^{10}(n)$
10	5	44	38	77	71	11	5	0	0	0
11	5	44	38	77	71	11	0	0	0	0
12	15	45	48	78	81	12	0	0	0	0
13	25	46	58	79	91	13	0	0	0	0
14	35	47	68	80	2	14	0	0	0	0
15	45	48	78	81	12	15	0	0	0	0
16	55	49	88	82	22	16	0	0	0	0
17	65	50	1	4	34	37	67	70	1	0
18	75	51	9	84	42	18	0	0	0	0
19	85	52	19	0	0	0	0	0	0	0
20	4	34	37	67	70	1	4	0	0	0
21	6	54	39	87	72	21	0	0	0	0
22	16	55	49	88	82	22	0	0	0	0
23	26	56	59	89	92	23	0	0	0	0
24	36	57	69	90	3	24	0	0	0	0
25	46	58	79	91	13	25	0	0	0	0
26	56	59	89	92	23	26	0	0	0	0

Continued on next page

n	$f(n)$	$f^2(n)$	$f^3(n)$	$f^4(n)$	$f^5(n)$	$f^6(n)$	$f^7(n)$	$f^8(n)$	$f^9(n)$	$f^{10}(n)$
27	66	60	60	0	0	0	0	0	0	0
28	76	61	10	5	44	38	77	71	11	5
29	86	62	20	4	34	37	67	70	1	4
30	3	24	36	57	69	90	3	0	0	0
31	7	64	40	2	14	35	47	68	80	2
32	17	65	50	1	4	34	37	67	70	1
33	27	66	60	60	0	0	0	0	0	0
34	37	67	70	1	4	34	0	0	0	0
35	47	68	80	2	14	35	0	0	0	0

where $f(n) = \left|n' - c\right|$, n' is digital reverse, $c = 6$ and $f^k(n) = f(f(\ldots(f(n))))$ of k times.

3.18.3 The Multiplication Periodic Sequences

Let $c > 1$ be a positive integer. Start with a positive integer n, multiply each digit x of n by c and replace that digit by the last digit of $c \cdot x$ to give n_1. And so on. We shall eventually obtain a repetition.

For example, with $c = 7$ and $n = 68$ we obtain the sequence: 68, 26, 42, 84, 68. Integers whose digits are all equal to 5 are invariant under the given operation after one iteration.

One studies the one–*digit multiplication periodic sequences* (short *dmps*) only. For c of two or more digits the problem becomes more complicated.

Program 3.43. for the *dmps*.

$$MPS(n,c) := \left| \begin{array}{l} d \leftarrow reverse(dn(n,10)) \\ m \leftarrow 0 \\ for\ k \in 1..last(d) \\ \quad m \leftarrow m + \mod(d_k \cdot c, 10) \cdot 10^{k-1} \\ return\ m \end{array} \right.$$

PPS program execution to vector (68) is

$$PPS((68), 7, 10, MPS) = (68\ \ 26\ \ 42\ \ 84\ \ 68)\ .$$

For example we use commands: $k := 10..19$ $v_k := k$.

1. If $c := 2$, there are four term loops, starting on the first or second term and $PPS(v, c, 10, MPS) =$:

Table 3.42: 2–*dmps* with $c = 2$

n	$f(n)$	$f^2(n)$	$f^3(n)$	$f^4(n)$	$f^5(n)$
10	20	40	80	60	20
11	22	44	88	66	22
12	24	48	86	62	24
13	26	42	84	68	26
14	28	46	82	64	28
15	20	40	80	60	20
16	22	44	88	66	22
17	24	48	86	62	24
18	26	42	84	68	26
19	28	46	82	64	28

where $f(n) = MPS(n,2)$ and $f^k(n) = f(f(\ldots(f(n))))$ of k times.

2. If $c := 3$, there are four term loops, starting with the first term and $PPS(v, c, 10, MPS) =$:

Table 3.43: 2–*dmps* with $c = 3$

n	$f(n)$	$f^2(n)$	$f^3(n)$	$f^4(n)$
10	30	90	70	10
11	33	99	77	11
12	36	98	74	12
13	39	97	71	13
14	32	96	78	14
15	35	95	75	15
16	38	94	72	16
17	31	93	79	17
18	34	92	76	18
19	37	91	73	19

where $f(n) = MPS(n,3)$ and $f^k(n) = f(f(\ldots(f(n))))$ of k times.

3. If $c := 4$, there are two term loops, starting on the first or second term (could be called switch or pendulum) and $PPS(v, c, 10, MPS) =$:

Table 3.44: 2–*dmps* with $c = 4$

n	$f(n)$	$f^2(n)$	$f^3(n)$
10	40	60	40
11	44	66	44
12	48	62	48
13	42	68	42
14	46	64	46
15	40	60	40
16	44	66	44
17	48	62	48
18	42	68	42
19	46	64	46

where $f(n) = MPS(n,4)$ and $f^k(n) = f(f(\ldots(f(n))))$ of k times.

4. If $c := 5$, the sequence is invariant after one iteration and $PPS(v, c, 10, MPS) =$:

Table 3.45: 2–*dmps* with $c = 5$

n	$f(n)$	$f^2(n)$
10	50	50
11	55	55
12	50	50
13	55	55
14	50	50
15	55	55
16	50	50
17	55	55
18	50	50
19	55	55

where $f(n) = MPS(n,5)$ and $f^k(n) = f(f(\ldots(f(n))))$ of k times.

5. If $c := 6$, the sequence is invariant after one iteration and $PPS(v, c, 10, MPS) =$:.

Table 3.46: 2–*dmps* with $c = 6$

n	$f(n)$	$f^2(n)$
10	60	60
11	66	66
12	62	62
13	68	68
14	64	64
15	60	60
16	66	66
17	62	62
18	68	68
19	64	64

where $f(n) = MPS(n, 6)$ and $f^k(n) = f(f(\ldots(f(n))))$ of k times.

6. If $c := 7$, there are four term loops, starting with the first term and $PPS(v, c, 10, MPS) =$:

Table 3.47: 2–*dmps* with $c = 7$

n	$f(n)$	$f^2(n)$	$f^3(n)$	$f^4(n)$
10	70	90	30	10
11	77	99	33	11
12	74	98	36	12
13	71	97	39	13
14	78	96	32	14
15	75	95	35	15
16	72	94	38	16
17	79	93	31	17
18	76	92	34	18
19	73	91	37	19

where $f(n) = MPS(n, 7)$ and $f^k(n) = f(f(\ldots(f(n))))$ of k times.

7. If $c := 8$, there are four term loops, starting on the first or second term and $PPS(v, c, 10, MPS) =$:

Table 3.48: 2–*dmps* with $c = 8$

n	$f(n)$	$f^2(n)$	$f^3(n)$	$f^4(n)$	$f^5(n)$
10	80	40	20	60	80
11	88	44	22	66	88
12	86	48	24	62	86
13	84	42	26	68	84
14	82	46	28	64	82
15	80	40	20	60	80
16	88	44	22	66	88
17	86	48	24	62	86
18	84	42	26	68	84
19	82	46	28	64	82

where $f(n) = MPS(n,8)$ and $f^k(n) = f(f(\ldots(f(n))))$ of k times.

8. If $c := 9$, there are two term loops, starting with the first term (pendulum) and $PPS(v,c,10,MPS) =$:

Table 3.49: 2–*dmps* with $c = 9$

n	$f(n)$	$f^2(n)$
10	90	10
11	99	11
12	98	12
13	97	13
14	96	14
15	95	15
16	94	16
17	93	17
18	92	18
19	91	19

where $f(n) = MPS(n,9)$ and $f^k(n) = f(f(\ldots(f(n))))$ of k times.

3.18.4 The Mixed Composition Periodic Sequences

Let n be a two-digits number. Add the digits, and add them again if the sum is greater than 10. Also take the absolute value of the difference of their digits. These are the first and second digits of n_1. Now repeat this.

Program 3.44. for the mixed composition periodic sequences.

$$MixPS(n,c) := \left| \begin{array}{l} d \leftarrow dn(n,10) \\ a_2 \leftarrow \sum d \\ \sum dn(a_2,10) \;\; if \;\; a_2 > 9 \\ a_1 \leftarrow |d_2 - d_1| \\ return \;\; a_2 \cdot 10 + a_1 \end{array} \right.$$

where used program *dn*, 2.2 and the function Mathcad $\sum v$, which sums vector elements v.

For example, with $n = 75$ we obtain the sequence:

$$PPS[(75),0,20,MixPS] = \\ (75\ \ 32\ \ 51\ \ 64\ \ 12\ \ 31\ \ 42\ \ 62\ \ 84\ \ 34\ \ 71 \\ 86\ \ 52\ \ 73\ \ 14\ \ 53\ \ 82\ \ 16\ \ 75)\ ,$$

(variable c has no role, i.e. can be $c = 0$).

There are no invariants in this case. Four numbers: 36, 90, 93, and 99 produce two-element loops. The longest loops have 18 elements. There also are loops of 4, 6, and 12 elements, [Ibstedt, 1997].

The mixed composition periodic sequences obtained with commands; $k := 5..25$ $p_{k-4} := prime_k$ $PPS(p,0,21,MixPS) =$

$$\begin{bmatrix}
11 & 20 & 22 & 40 & 44 & 80 & 88 & 70 & 77 & 50 & 55 & 10 & 11 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
13 & 42 & 62 & 84 & 34 & 71 & 86 & 52 & 73 & 14 & 53 & 82 & 16 & 75 & 32 & 51 & 64 & 12 & 31 & 42 & 0 \\
17 & 86 & 52 & 73 & 14 & 53 & 82 & 16 & 75 & 32 & 51 & 64 & 12 & 31 & 42 & 62 & 84 & 34 & 71 & 86 & 0 \\
19 & 18 & 97 & 72 & 95 & 54 & 91 & 18 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
23 & 51 & 64 & 12 & 31 & 42 & 62 & 84 & 34 & 71 & 86 & 52 & 73 & 14 & 53 & 82 & 16 & 75 & 32 & 51 & 0 \\
29 & 27 & 95 & 54 & 91 & 18 & 97 & 72 & 95 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
31 & 42 & 62 & 84 & 34 & 71 & 86 & 52 & 73 & 14 & 53 & 82 & 16 & 75 & 32 & 51 & 64 & 12 & 31 & 0 & 0 \\
37 & 14 & 53 & 82 & 16 & 75 & 32 & 51 & 64 & 12 & 31 & 42 & 62 & 84 & 34 & 71 & 86 & 52 & 73 & 14 & 0 \\
41 & 53 & 82 & 16 & 75 & 32 & 51 & 64 & 12 & 31 & 42 & 62 & 84 & 34 & 71 & 86 & 52 & 73 & 14 & 53 & 0 \\
43 & 71 & 86 & 52 & 73 & 14 & 53 & 82 & 16 & 75 & 32 & 51 & 64 & 12 & 31 & 42 & 62 & 84 & 34 & 71 & 0 \\
47 & 23 & 51 & 64 & 12 & 31 & 42 & 62 & 84 & 34 & 71 & 86 & 52 & 73 & 14 & 53 & 82 & 16 & 75 & 32 & 51 \\
53 & 82 & 16 & 75 & 32 & 51 & 64 & 12 & 31 & 42 & 62 & 84 & 34 & 71 & 86 & 52 & 73 & 14 & 53 & 0 & 0 \\
59 & 54 & 91 & 18 & 97 & 72 & 95 & 54 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
61 & 75 & 32 & 51 & 64 & 12 & 31 & 42 & 62 & 84 & 34 & 71 & 86 & 52 & 73 & 14 & 53 & 82 & 16 & 75 & 0 \\
67 & 41 & 53 & 82 & 16 & 75 & 32 & 51 & 64 & 12 & 31 & 42 & 62 & 84 & 34 & 71 & 86 & 52 & 73 & 14 & 53 \\
71 & 86 & 52 & 73 & 14 & 53 & 82 & 16 & 75 & 32 & 51 & 64 & 12 & 31 & 42 & 62 & 84 & 34 & 71 & 0 & 0 \\
73 & 14 & 53 & 82 & 16 & 75 & 32 & 51 & 64 & 12 & 31 & 42 & 62 & 84 & 34 & 71 & 86 & 52 & 73 & 0 & 0 \\
79 & 72 & 95 & 54 & 91 & 18 & 97 & 72 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
83 & 25 & 73 & 14 & 53 & 82 & 16 & 75 & 32 & 51 & 64 & 12 & 31 & 42 & 62 & 84 & 34 & 71 & 86 & 52 & 73 \\
89 & 81 & 97 & 72 & 95 & 54 & 91 & 18 & 97 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
97 & 72 & 95 & 54 & 91 & 18 & 97 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

3.18.5 Kaprekar Periodic Sequences

Kaprekar proposed the following algorithm:

Algorithm 3.45. Fie $n \in \mathbb{N}^*$, we sort the number digits n in decreasing order, thus the resulting number is n', we sort the number digits n in increasing order, thus the resulting number is n''. We denote by $K(n)$ the number $n' - n''$.

Let the function $K : \{1000, 1001, \ldots, 9999\}$,

Function 3.46. for the algorithm 3.45.

$$K(n) = reverse(sort(dn(n,10))) \cdot Vb(10, nrd(n,10)) \\ - sort(dn(n,10)) \cdot Vb(10, nrd(n,10)) \ , \quad (3.18)$$

where function *dn*, 2.2, gives the vector with digits of numbers in the indicated numeration base and the Mathcad functions: *sort* for ascending sorting of a vector and *reverse* for reading a vector from tail to head. Function Vb provides the vector $(10^{m-1} \ \ 10^{m-2} \ \ \ldots \ \ 1)^{\mathrm{T}}$, where $m = nrd(n,10)$, i.e. the digits number of the number n in base 10 10.

Examples: $K(7675) = 2088$, $K(3215) = 4086$, $K(5107) = 7353$.

Since 1949 Kaprekar noted that if we apply several times to any number with four digits the above algorithm, we get the number 6147. Kaprekar [1955] conjectured that $K^m(n) = 6147$ for $m \le 7$ si $n \ne 1111, 2222, \ldots, 9999$, the number 6147 is called *Kaprekar constant*. The studies that followed, [Deutsch and Goldman, 2004, Weisstein, 2015a], confirmed the Kaprekar conjecture. For exemplification, we consider the first 27 primes with 4 digits using the controls $k := 169..195$, $p_{k-168} := prime_k$, then by the call $PPS(p, 8, K) =$ we obtain the matrix:

Table 3.50: 4–digits Kaprekar periodic sequences

n	$K(n)$	$K^2(n)$	$K^3(n)$	$K^4(n)$	$K^5(n)$	$K^6(n)$	$K^7(n)$
1009	9081	9621	8352	6174	6174	0	0
1013	2997	7173	6354	3087	8352	6174	6174
1019	8991	8082	8532	6174	6174	0	0
1021	1998	8082	8532	6174	6174	0	0
1031	2997	7173	6354	3087	8352	6174	6174
1033	3177	6354	3087	8352	6174	6174	0
1039	9171	8532	6174	6174	0	0	0
1049	9261	8352	6174	6174	0	0	0
1051	4995	5355	1998	8082	8532	6174	6174

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n	$K(n)$	$K^2(n)$	$K^3(n)$	$K^4(n)$	$K^5(n)$	$K^6(n)$	$K^7(n)$
1061	5994	5355	1998	8082	8532	6174	6174
1063	6174	6174	0	0	0	0	0
1069	9441	7992	7173	6354	3087	8352	6174
1087	8532	6174	6174	0	0	0	0
1091	8991	8082	8532	6174	6174	0	0
1093	9171	8532	6174	6174	0	0	0
1097	9531	8172	7443	3996	6264	4176	6174
1103	2997	7173	6354	3087	8352	6174	6174
1109	8991	8082	8532	6174	6174	0	0
1117	5994	5355	1998	8082	8532	6174	6174
1123	2088	8532	6174	6174	0	0	0
1129	8082	8532	6174	6174	0	0	0
1151	3996	6264	4176	6174	6174	0	0
1153	4176	6174	6174	0	0	0	0
1163	5175	5994	5355	1998	8082	8532	6174
1171	5994	5355	1998	8082	8532	6174	6174
1181	6993	6264	4176	6174	6174	0	0
1187	7533	4176	6174	6174	0	0	0

- For numbers with 2–digits. Applying the most 7 simple iteration, i.e. $n = K(n)$, the Kaprekar algorithm becomes equal to one of *Kaprekar constants*, and in these values the function K becomes periodical of periods 1 or 5, as seen in the following Table:

CK	ν	p
0	9	1
$9 = 3^2$	24	5
$27 = 3^3$	24	5
$45 = 3^2 \cdot 5$	12	5
$63 = 3^2 \cdot 7$	20	5
$81 = 3^4$	1	5

  where CK = *Kaprekar constants*, ν = *frequent* and p = *periodicity*. It follows that is fixed point for the algorithm 3.45 with frequency of 9 times (for $11, 22, \ldots, 99$) with periodicity 1, i.e. $K(0) = 0$; 9 is fixed point for function K with frequency of 24 times and with periodicity 5, i.e. $K^5(9) = 9$; $\ldots$.

- For numbers with 3–digits. Applying the most 7 simple iteration, i.e. $n := K(n)$, the algorithm 3.45 becomes equal to one of the *Kaprekar constants*

from the following Table:

CK	v	p
0	9	1
$495 = 3^2 \cdot 5 \cdot 11$	891	1

and in these values the function K becomes periodical of period 1, i.e. $K(CK) = CK$, it follows that 0 and 495 are fixed point for the function K.

- For numbers with 4–digits. Applying the most 7 simple iteration, i.e. $n := K(n)$, the algorithm 3.45 becomes equal to one of the *Kaprekar constants* from the following Table:

CK	v	p
0	9	1
$6174 = 2 \cdot 3^2 \cdot 7^3$	8991	1

and in these values the function K becomes periodical of period 1, i.e. $K(CK) = CK$, it follows that 0 and 6174 are fixed point for the function K.

- For numbers with 5–digits. Applying the most 67 simple iteration, i.e. $n := K(n)$, $K(n)$ becomes equal to one of *Kaprekar constants* from the following Table:

CK	v	p
0	9	1
$53955 = 3^2 \cdot 5 \cdot 11 \cdot 109$	2587	2
$59994 = 2 \cdot 3^3 \cdot 11 \cdot 101$	415	2
$61974 = 2 \cdot 3^2 \cdot 11 \cdot 313$	4770	4
$62964 = 2^2 \cdot 3^3 \cdot 11 \cdot 53$	4754	4
$63954 = 2 \cdot 3^2 \cdot 11 \cdot 17 \cdot 19$	24164	4
$71973 = 3^2 \cdot 11 \cdot 727$	5816	4
$74943 = 3^2 \cdot 11 \cdot 757$	27809	4
$75933 = 3^2 \cdot 11 \cdot 13 \cdot 59$	9028	4
$82962 = 2 \cdot 3^2 \cdot 11 \cdot 419$	5808	4
$83952 = 2^4 \cdot 3^2 \cdot 11 \cdot 53$	4840	4

and in these values the function K becomes periodical of periods of 1, 2 or 4. It follows that 0 is fixed point for 3.45, i.e. $K(0) = 0$, 53955 and 59994 are of periods 2, equivalent to $K^2(CK) = K(K(CK)) = CK$, and the rest are of period 4, i.e. $K^4(CK) = CK$. We note that all CK are multiples of $3^2 \cdot 11 = 99$.

- For numbers with 6–digits. Applying the most 11 simple iteration, i.e. $n := K(n)$, the function K becomes equal to one of the *Kaprekar constants* from the following Table:

CK	v	p
0	4	1
$4420876 = 2^2 * 3^5 \cdot 433$	154591	7
$549945 = 3^2 \cdot 5 \cdot 11^2 \cdot 101$	840	1
$631764 = 2^2 \cdot 3^2 \cdot 7 \cdot 23 \cdot 109$	24920	1
$642654 = 2 \cdot 3^4 \cdot 3967$	13050	7
$750843 = 3^3 \cdot 27809$	15845	7
$840852 = 2^2 \cdot 3^2 \cdot 23357$	24370	7
$851742 = 2 \cdot 3^3 \cdot 15773$	101550	7
$860832 = 2^5 \cdot 3^2 \cdot 7^2 \cdot 61$	51730	7
$862632 = 2^3 \cdot 3^2 \cdot 11981$	13100	7

  and in these values the function K becomes periodical of periods 1 or 7. It follows that 0, 549945 and 631764 are fixed points for K, and the rest are of periods 7, i.e. $K^7(CK) = CK$.

3.18.6 The Permutation Periodic Sequences

A generalization of the regular functions would be the function resulting from the following algorithm

Algorthm 3.47. Let $n \in \mathbb{N}^*$, be a number with m digits, i.e. $n = \overline{d_1 d_2 \ldots d_m}$. We consider a permutation of the set $\{1, 2, \ldots, m\}$, $pr = (i_1, i_2, \ldots, i_m)$, then the number n' is given by the digits permutations of the numb n using the permutation pr, $n' = \overline{d_{i_1} d_{i_2} \ldots d_{i_m}}$. The new number equals $\left|n - n'\right|$.

Program 3.48. for the algorithm 3.47.

$$PSP(n, pr) := \left| \begin{array}{l} d \leftarrow dn(n, 10, last(pr)) \\ for\ k \in 1..last(d) \\ \quad nd_k \leftarrow d_{(pr_k)} \\ return\ |n - nd \cdot Vb(10, last(d))| \end{array} \right.$$

For example, with 2 digits sequence by commands $k := 1..27$ $v_k := 13 + k$ $PPS\big(v, 14, (2\ \ 1)^{\mathrm{T}}\big) =$ generates the matrix:

Table 3.51: 2–digits permutation periodic sequences

n	$f(n)$	$f^2(n)$	$f^3(n)$	$f^4(n)$	$f^5(n)$	$f^6(n)$	$f^7(n)$	$f^8(n)$
14	27	45	9	81	63	27	0	5
15	36	27	45	9	81	63	27	5
16	45	9	81	63	27	45	0	5
17	54	9	81	63	27	45	9	5
18	63	27	45	9	81	63	0	5
19	72	45	9	81	63	27	45	5
20	18	63	27	45	9	81	63	5
21	9	81	63	27	45	9	0	5
22	22	0	0	0	0	0	0	1
23	9	81	63	27	45	9	0	5
24	18	63	27	45	9	81	63	5
25	27	45	9	81	63	27	0	5
26	36	27	45	9	81	63	27	5
27	45	9	81	63	27	0	0	5
28	54	9	81	63	27	45	9	5
29	63	27	45	9	81	63	0	5
30	27	45	9	81	63	27	0	5
31	18	63	27	45	9	81	63	5
32	9	81	63	27	45	9	0	5
33	33	0	0	0	0	0	0	1
34	9	81	63	27	45	9	0	5
35	18	63	27	45	9	81	63	5
36	27	45	9	81	63	27	0	5
37	36	27	45	9	81	63	27	5
38	45	9	81	63	27	45	0	5
39	54	9	81	63	27	45	9	5
40	36	27	45	9	81	63	27	5

Analysis tells us that this matrix of *constants periodic sequences* associated to the permutation $(2\ \ 1)^{\mathrm{T}}$ are the nonnull penultimate numbers in each row of the matrix: 27, 45, 63, 9, 22, 33 (in order of appearance in the matrix). The last column of the matrix represents the periodicity of each *constant periodic sequences*. The constants 27, 45, 63 and 9 have the periodicity 5, i.e. $PSP^5(27) = 27$, $PSP^5(45) = 45$, etc. The constants 22 and 33 have a periodicity equals to 1, i.e. $PSP(22) = 0$ and $PSP(33) = 0$. One may count the frequency of occurrence of each *constant periodic sequences*.

We present a study on permutation periodic sequences for 3–digits numbers having 3 digits relatively to the 6th permutation of the set $\{1,2,3\}$. The required commands are: $k := 100..999$ $v_{k-99} := k$ $PPS\big(v,20,pr\big) =$, where pr is a permutation of the set $\{1,2,3\}$.

1. For the permutation $(2\ 3\ 1)^{\mathrm{T}}$, the commands $k := 970..999$ $v_{k-969} := k$ $PPS\big(v,20,(2\ 3\ 1)^{\mathrm{T}}\big) =$ generates the matrix:

Table 3.52: 3–digits PPS with permutation $(2\ 3\ 1)^{\mathrm{T}}$

n	$f(n)$	$f^2(n)$	$f^3(n)$	$f^4(n)$	$f^5(n)$	$f^6(n)$	$f^7(n)$	$f^8(n)$	$f^9(n)$	$f^{10}(n)$	$f^{11}(n)$	$f^{12}(n)$
970	261	351	162	459	135	216	54	486	378	405	351	0
971	252	270	432	108	27	243	189	702	675	81	729	432
972	243	189	702	675	81	729	432	108	27	243	0	0
973	234	108	27	243	189	702	675	81	729	432	108	0
974	225	27	243	189	702	675	81	729	432	108	27	0
975	216	54	486	378	405	351	162	459	135	216	0	0
976	207	135	216	54	486	378	405	351	162	459	135	0
977	198	783	54	486	378	405	351	162	459	135	216	54
978	189	702	675	81	729	432	108	27	243	189	0	0
979	180	621	405	351	162	459	135	216	54	486	378	405
980	171	540	135	216	54	486	378	405	351	162	459	135
981	162	459	135	216	54	486	378	405	351	162	0	0
982	153	378	405	351	162	459	135	216	54	486	378	0
983	144	297	675	81	729	432	108	27	243	189	702	675
984	135	216	54	486	378	405	351	162	459	135	0	0
985	126	135	216	54	486	378	405	351	162	459	135	0
986	117	54	486	378	405	351	162	459	135	216	54	0
987	108	27	243	189	702	675	81	729	432	108	0	0
988	99	891	27	243	189	702	675	81	729	432	108	27
989	90	810	702	675	81	729	432	108	27	243	189	702
990	81	729	432	108	27	243	189	702	675	81	0	0
991	72	648	162	459	135	216	54	486	378	405	351	162
992	63	567	108	27	243	189	702	675	81	729	432	108
993	54	486	378	405	351	162	459	135	216	54	0	0
994	45	405	351	162	459	135	216	54	486	378	405	0
995	36	324	81	729	432	108	27	243	189	702	675	81
996	27	243	189	702	675	81	729	432	108	27	0	0
997	18	162	459	135	216	54	486	378	405	351	162	0
998	9	81	729	432	108	27	243	189	702	675	81	0
999	0	0	0	0	0	0	0	0	0	0	0	0

We have the following list of *constant periodic sequences*, with frequency of occurrence ν and periodicity p.

Table 3.53: 3–digits PPS with the permutation $(2\ 3\ 1)^{\mathrm{T}}$

CPS	ν	p
0	9	1

Continued on next page

CPS	ν	p
$27 = 3^3$	67	9
$54 = 2 \cdot 3^3$	68	9
$81 = 3^4$	94	9
$108 = 2^2 \cdot 3^3$	71	9
$135 = 3^3 \cdot 5$	70	9
$162 = 2 \cdot 3^4$	70	9
$189 = 3^3 \cdot 7$	34	9
$216 = 2^3 \cdot 3^3$	30	9
$243 = 3^5$	45	9
$333 = 3^2 \cdot 37$	11	1
$351 = 3^3 \cdot 13$	50	9
$378 = 2 \cdot 3^3 \cdot 7$	50	9
$405 = 3^4 \cdot 5$	47	9
$432 = 2^4 \cdot 3^3$	54	9
$459 = 3^3 \cdot 17$	26	9
$486 = 2 \cdot 3^5$	25	9
$666 = 2 \cdot 3^2 \cdot 37$	5	1
$675 = 3^3 \cdot 5^2$	50	9
$702 = 2 \cdot 3^3 \cdot 13$	21	9
$729 = 3^6$	3	9

We note that all nonnull *constants periodic sequences* are multiples of 3^2.

2. For permutation $(3\ 1\ 2)^T$ we have the following list of *constant periodic sequences*, with frequency of occurrence ν and periodicity p.

Table 3.54: 3–digits PPS with the permutation $(3\ 1\ 2)^T$

CPS	ν	p
0	9	1
$27 = 3^3$	48	9
$54 = 2 \cdot 3^3$	50	9
$81 = 3^4$	52	9
$108 = 2^2 \cdot 3^3$	76	9
$135 = 3^3 \cdot 5$	70	9
$162 = 2 \cdot 3^4$	71	9
$189 = 3^3 \cdot 7$	70	9

Continued on next page

CPS	v	p
$216 = 2^3 \cdot 3^3$	50	9
$243 = 3^5$	91	9
$333 = 3^2 \cdot 37$	11	1
$351 = 3^3 \cdot 13$	27	9
$378 = 2 \cdot 3^3 \cdot 7$	28	9
$405 = 3^4 \cdot 5$	45	9
$432 = 2^4 \cdot 3^3$	54	9
$459 = 3^3 \cdot 17$	47	9
$486 = 2 \cdot 3^5$	48	9
$666 = 2 \cdot 3^2 \cdot 37$	5	1
$675 = 3^3 \cdot 5^2$	5	9
$702 = 2 \cdot 3^3 \cdot 13$	22	9
$729 = 3^6$	21	9

We note that we have the same Kaprecar constants as in permutation $(2\ 3\ 1)^{\mathrm{T}}$ just that there are other frequency of occurrence.

3. For permutation $(1\ 3\ 2)^{\mathrm{T}}$, the commands $k := 300..330$ $v_{k-299} := k$ $PPS\big(v, 20, (1\ 3\ 2)^{\mathrm{T}}\big) =$ generates the matrix:

Table 3.55: 3–digits PPS with permutation $(1\ 3\ 2)^{\mathrm{T}}$

n	$f(n)$	$f^2(n)$	$f^3(n)$	$f^4(n)$	$f^5(n)$	$f^6(n)$	$f^7(n)$
300	0	0	0	0	0	0	0
301	9	81	63	27	45	9	0
302	18	63	27	45	9	81	63
303	27	45	9	81	63	27	0
304	36	27	45	9	81	63	27
305	45	9	81	63	27	45	0
306	54	9	81	63	27	45	9
307	63	27	45	9	81	63	0
308	72	45	9	81	63	27	45
309	81	63	27	45	9	81	0
310	9	81	63	27	45	9	0
311	0	0	0	0	0	0	0
312	9	81	63	27	45	9	0
313	18	63	27	45	9	81	63

Continued on next page

n	$f(n)$	$f^2(n)$	$f^3(n)$	$f^4(n)$	$f^5(n)$	$f^6(n)$	$f^7(n)$
314	27	45	9	81	63	27	0
315	36	27	45	9	81	63	27
316	45	9	81	63	27	45	0
317	54	9	81	63	27	45	9
318	63	27	45	9	81	63	0
319	72	45	9	81	63	27	45
320	18	63	27	45	9	81	63
321	9	81	63	27	45	9	0
322	0	0	0	0	0	0	0
323	9	81	63	27	45	9	0
324	18	63	27	45	9	81	63
325	27	45	9	81	63	27	0
326	36	27	45	9	81	63	27
327	45	9	81	63	27	45	0
328	54	9	81	63	27	45	9
329	63	27	45	9	81	63	0
330	27	45	9	81	63	27	0

We have the following list of *constant periodic sequences*, with frequency of occurrences v and periodicity p.

CPS	v	p
0	90	1
$9 = 3^2$	234	5
$27 = 3^3$	234	5
$45 = 3^2 \cdot 5$	126	5
$63 = 3^2 \cdot 7$	198	5
$81 = 3^4$	18	5

4. For permutation $(2\ 1\ 3)^{\mathrm{T}}$ we have the following list of *constant periodic sequences*, with frequency of occurrences v and periodicity p.

CPS	v	p
0	90	1
$90 = 2 \cdot 3^2 \cdot 5$	240	5
$270 = 2 \cdot 3^3 \cdot 5$	240	5
$450 = 2 \cdot 3^2 \cdot 5^2$	120	5
$630 = 2 \cdot 5^2 \cdot 5 \cdot 7$	200	5
$810 = 2 \cdot 3^4 \cdot 5$	10	5

5. For permutation $(3\ 2\ 1)^{\mathrm{T}}$ we have the following list of *constant periodic sequences*, with frequency of occurrences ν and periodicity p.

CPS	ν	p
0	90	1
$99 = 3^2 \cdot 11$	240	5
$297 = 3^3 \cdot 11$	240	5
$495 = 3^2 \cdot 5 \cdot 11$	120	5
$693 = 3^2 \cdot 7 \cdot 11$	200	5
$891 = 3^4 \cdot 11$	10	5

6. For identical permutation, obvious, we have only *constant periodic sequences* 0 with frequency 900 and periodicity 1.

3.19 Erdös–Smarandache Numbers

The solutions to the Diophantine equation $P(n) = S(n)$, where $P(n)$ is the largest prime factor which divides n, and $S(n)$ is the classical Smarandache function, are Erdös–Smarandache numbers, [Erdös and Ashbacher, 1997, Tabirca, 2004], [Sloane, 2014, A048839].

Program 3.49. generation the Erdös–Smarandache numbers.

$$
ES(a,b) := \left|
\begin{array}{l}
j \leftarrow 0 \\
for\ n \in a..b \\
\quad \left|
\begin{array}{l}
m \leftarrow max\left(Fa(n)^{\langle 1 \rangle}\right) \\
if\ S(n)=m \\
\quad \left|
\begin{array}{l}
j \leftarrow j+1 \\
s_j \leftarrow n
\end{array}
\right.
\end{array}
\right. \\
return\ \ s
\end{array}
\right.
$$

The program use S (Smarandache function) and Fa (of factorization the numbers) programs.

Erdös–Smarandache numbers one obtains with the command $ES(2,200)^{\mathrm{T}} =$ 2, 3, 5, 6, 7, 10, 11, 13, 14, 15, 17, 19, 20, 21, 22, 23, 26, 28, 29, 30, 31, 33, 34, 35, 37, 38, 39, 40, 41, 42, 43, 44, 46, 47, 51, 52, 53, 55, 56, 57, 58, 59, 60, 61, 62, 63, 65, 66, 67, 68, 69, 70, 71, 73, 74, 76, 77, 78, 79, 82, 83, 84, 85, 86, 87, 88, 89, 91, 92, 93, 94, 95, 97, 99, 101, 102, 103, 104, 105, 106, 107, 109, 110, 111, 112, 113, 114, 115, 116, 117, 118, 119, 120, 122, 123, 124, 126, 127, 129, 130, 131, 132, 133, 134, 136, 137, 138, 139, 140, 141, 142, 143, 145, 146, 148, 149, 151, 152, 153, 154, 155, 156,

157, 158, 159, 161, 163, 164, 165, 166, 167, 168, 170, 171, 172, 173, 174, 176, 177, 178, 179, 181, 182, 183, 184, 185, 186, 187, 188, 190, 191, 193, 194, 195, 197, 198, 199 .

Program 3.50. generation the Erdös–Smarandache numbers not prime.

$$
\mathit{ES1}(a,b) := \left|
\begin{array}{l}
j \leftarrow 0 \\
for\ n \in a..b \\
\quad \left|
\begin{array}{l}
m \leftarrow max\left(Fa(n)^{\langle 1 \rangle}\right) \\
if\ S(n)=m \wedge S(n) \neq n \\
\quad \left|
\begin{array}{l}
j \leftarrow j+1 \\
s_j \leftarrow n
\end{array}
\right.
\end{array}
\right. \\
return\ \ s
\end{array}
\right.
$$

The program use S (Smarandache function) and Fa (of factorization the numbers) programs.

Erdös–Smarandache numbers, that not are primes, one obtains with command $ES1(2,200)^{\mathrm{T}}$ = 6, 10, 14, 15, 20, 21, 22, 26, 28, 30, 33, 34, 35, 38, 39, 40, 42, 44, 46, 51, 52, 55, 56, 57, 58, 60, 62, 63, 65, 66, 68, 69, 70, 74, 76, 77, 78, 82, 84, 85, 86, 87, 88, 91, 92, 93, 94, 95, 99, 102, 104, 105, 106, 110, 111, 112, 114, 115, 116, 117, 118, 119, 120, 122, 123, 124, 126, 129, 130, 132, 133, 134, 136, 138, 140, 141, 142, 143, 145, 146, 148, 152, 153, 154, 155, 156, 158, 159, 161, 164, 165, 166, 168, 170, 171, 172, 174, 176, 177, 178, 182, 183, 184, 185, 186, 187, 188, 190, 194, 195, 198 .

Program 3.51. generation the Erdös–Smarandache with *nf* prime factors.

$$
\mathit{ES2}(a,b,nf) := \left|
\begin{array}{l}
j \leftarrow 0 \\
for\ n \in a..b \\
\quad \left|
\begin{array}{l}
m \leftarrow max\left(Fa(n)^{\langle 1 \rangle}\right) \\
if\ S(n)=m \wedge rows\left(Fa(n)^{\langle 1 \rangle}\right)=nf \\
\quad \left|
\begin{array}{l}
j \leftarrow j+1 \\
s_j \leftarrow n
\end{array}
\right.
\end{array}
\right. \\
return\ \ s
\end{array}
\right.
$$

The program use S (Smarandache function) and Fa (of factorization the numbers) programs.

1. Erdös–Smarandache numbers, that have exactly two prime factors, one obtains with the command $ES2(2,200,2)^{\mathrm{T}}$ = 6, 10, 14, 15, 20, 21, 22, 26, 28, 33, 34, 35, 38, 39, 40, 44, 46, 51, 52, 55, 56, 57, 58, 62, 63, 65, 68, 69, 74,

76, 77, 82, 85, 86, 87, 88, 91, 92, 93, 94, 95, 99, 104, 106, 111, 112, 115, 116, 117, 118, 119, 122, 123, 124, 129, 133, 134, 136, 141, 142, 143, 145, 146, 148, 152, 153, 155, 158, 159, 161, 164, 166, 171, 172, 176, 177, 178, 183, 184, 185, 187, 188, 194 .

2. Erdös–Smarandache numbers, that have exactly three prime factors, one obtains with the command $ES2(2,200,3)^{\mathrm{T}}$ = 30, 42, 60, 66, 70, 78, 84, 102, 105, 110, 114, 120, 126, 130, 132, 138, 140, 154, 156, 165, 168, 170, 174, 182, 186, 190, 195, 198 .

3. Erdös–Smarandache numbers, that have exactly four prime factors, one obtains with the command $ES2(2,1000,4)^{\mathrm{T}}$ = 210, 330, 390, 420, 462, 510, 546, 570, 630, 660, 690, 714, 770, 780, 798, 840, 858, 870, 910, 924, 930, 966, 990 .

4. Erdös–Smarandache numbers, that have exactly five prime factors, one obtains with the command $ES2(2,5000,5)^{\mathrm{T}}$ = 2310, 2730, 3570, 3990, 4290, 4620, 4830 .

3.20 Multiplicative Sequences

3.20.1 Multiplicative Sequence of First 2 Terms

General definition: if m_1, m_2, are the first two terms of the sequence, then m_k, for $k \geq 3$, is the smallest number equal to the product of two previous distinct terms.

Program 3.52. generate multiplicative sequence with first two terms.

```
MS2(m1, m2, L) := | m ← (m1  m2)^T
                  | j ← 3
                  | while j ≤ L
                  |   | i ← 1
                  |   | for k1 ∈ 1..last(m) − 1
                  |   |   for k2 ∈ k1 + 1..last(m)
                  |   |     | y ← m_k1 · m_k2
                  |   |     | if y > max(m)
                  |   |     |   | x_i ← y
                  |   |     |   | i ← i + 1
                  |   | m_j ← min(x)
                  |   | j ← j + 1
                  | return m
```

Examples:

1. $ms23 := MS2(2,3,25)$ $ms23^{\mathrm{T}} \rightarrow (2\ 3\ 6\ 12\ 18\ 24\ 36\ 48\ 54\ 72\ 96\ 108\ 144\ 162\ 192\ 216\ 288\ 324\ 384\ 432\ 486\ 576\ 648\ 972\ 1458)$;

2. $ms37 := MS2(3,7,25)$ $ms37^{T} \rightarrow (3\ 7\ 21\ 63\ 147\ 189\ 441\ 567\ 1029\ 1323\ 1701\ 3087\ 3969\ 5103\ 7203\ 9261\ 11907\ 15309\ 21609\ 27783\ 35721\ 50421\ 64827\ 151263\ 352947)$;

3. $ms1113 := MS2(11,13,25)$ $ms1113^{\mathrm{T}} \rightarrow (11\ 13\ 143\ 1573\ 1859\ 17303\ 20449\ 24167\ 190333\ 224939\ 265837\ 314171\ 2093663\ 2474329\ 2924207\ 3455881\ 4084223\ 23030293\ 27217619\ 32166277\ 38014691)$.

The program *MS2*, 3.52, is equivalent with the program that generates the products m_1, m_2, $m_1 \cdot m_2$, $m_1^2 \cdot m_2^1$, $m_1^1 \cdot m_2^2$, $m_1^3 \cdot m_2^1$, $m_1^2 \cdot m_2^2$, $m_1^1 \cdot m_2^3, \ldots$, noting that the series finally have to be ascending sorted because we have no guarantee that such terms are generated in ascending order. The program for this algorithm is simpler and probably faster..

Program 3.53. generate multiplicative sequence with first two terms.

$$
\begin{aligned}
VarMS2(m_1, m_2, L) := \Big|\ & m \leftarrow (m_1\ \ m_2)^{\mathrm{T}} \\
& for\ \ k \in 1..ceil\left(\frac{\sqrt{8L+1}-1}{2}\right) \\
& \quad \Big|\ for\ \ i \in 1..k \\
& \quad \quad \Big|\ m \leftarrow stack[m, (m_1)^{k+1-i} \cdot (m_2)^{i}] \\
& \quad \quad \Big|\ return\ \ sort(m)\ \ if\ \ last(m) \le L
\end{aligned}
$$

3.20.2 Multiplicative Sequence of First 3 Terms

General definition: if m_1, m_2, m_3, are the first two terms of the sequence, then m_k, for $k \ge 4$, is the smallest number equal to the product of three previous distinct terms.

Program 3.54. generate multiplicative sequence with first three terms.

$$
\begin{aligned}
MS3(m_1, m_2, m_3, L) := \Big|\ & m \leftarrow (m_1\ \ m_2\ \ m_3)^{\mathrm{T}} \\
& j \leftarrow 4 \\
& while\ \ j \le L \\
& \quad \Big|\ i \leftarrow 1 \\
& \quad \Big|\ for\ \ k_1 \in 1..last(m) - 1 \\
& \quad \quad for\ \ k_2 \in k_1 + 1..last(m) \\
& \quad \quad \quad for\ \ k_3 \in k_2 + 1..last(m) \\
& \quad \quad \quad \quad \Big|\ y \leftarrow m_{k_1} \cdot m_{k_2}
\end{aligned}
$$

$$
\begin{array}{|l}
\quad\begin{array}{|l}
\qquad\quad\begin{array}{|l}
if\ \ y > max(m) \\
\begin{array}{|l}
x_i \leftarrow y \\
i \leftarrow i + 1
\end{array}
\end{array} \\
m_j \leftarrow min(x) \\
j \leftarrow j + 1
\end{array} \\
return\ \ m
\end{array}
$$

Examples:

1. *ms235* := *MS3*(2,3,5,30), $ms235^{\mathrm{T}}$ →(2, 3, 5, 30, 180, 300, 450, 1080, 1800, 2700, 3000, 4500, 6480, 6750, 10800, 16200, 18000, 27000, 30000, 38880, 40500, 45000, 64800, 67500, 97200, 101250, 108000, 162000, 180000, 233280) ;

2. *ms237* := *MS3*(2,3,7,30), $ms237^{\mathrm{T}}$ →(2, 3, 7, 42, 252, 588, 882, 1512, 3528, 5292, 8232, 9072, 12348, 18522, 21168, 31752, 49392, 54432, 74088, 111132, 115248, 127008, 172872, 190512, 259308, 296352, 326592, 388962, 444528, 666792) ;

3. *ms111317* := *MS3*(11,13,17,30), $ms111317^{\mathrm{T}}$ →(11, 13, 17, 2431, 347633, 454597, 537251, 49711519, 65007371, 76826893, 85009639, 100465937, 118732471, 7108747217, 9296054053, 10986245699, 12156378377, 14366628991, 15896802493, 16978743353, 18787130219, 22202972077, 26239876091, 1016550852031, 1329335729579, 1571033134957, 1738362107911, 2054427945713, 2273242756499, 2427960299479) .

One can write a program similar to the program *VarMS2*, 3.53, that genarete multiplicative sequence with first three terms.

3.20.3 Multiplicative Sequence of First k Terms

General definition: if m_1, m_2, ..., m_k are the first k terms of the sequence, then m_j, for $j \geq k+1$, is the smallest number equal to the product of k previous distinct terms.

3.21 Generalized Arithmetic Progression

A classic arithmetic progression is defined by: $a_1 \in \mathbb{R}$ the first term of progression, $r \neq 0$, $r \in \mathbb{R}$ ratio of progression (if $r > 0$ then we have an ascending progression, if $r < 0$ then we have a descending ratio), $a_{k+1} = a_k + r = a_1 + k \cdot r$ for any $k \in \mathbb{N}^*$, the term of rank $k+1$. Obviously, we can consider ascending

progressions of integers (or natural numbers) or descending progressions of integers (or natural numbers), where $a_1 \in \mathbb{Z}$, (or $a_1 \in \mathbb{N}$) $r \in \mathbb{Z}^*$ (or $r \in \mathbb{N}$) and $a_{k+1} = a_k + r = a_1 + k \cdot r$.

We consider the following generalization of arithmetic progressions. Let $a_1 \in \mathbb{R}$ the first term of the arithmetic progression and $\{r_k\} \subset \mathbb{R}$ series of positive numbers if ascending progressions and a series of negative numbers when descending progressions. This series we will call the series ratio. The term a_{k+1} is defined by formula:

$$a_{k+1} = a_k + r_k = a_1 + \sum_{j=1}^{k} r_j \ ,$$

for any $k \geq 1$.

Observation 3.55. It is obvious that a classical arithmetic progression is a particular case of the generalized arithmetic progression.

Examples (we preferred to give examples of progression of integers for easier reading of text):

1. Let $L := 35 \ \ k := 1..L \ \ r_k := 3$, then

   $r^{\mathrm{T}} \rightarrow$(3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3),

   if $a_1 := 2$ and $a_{k+1} := a_k + r_k$, then result

   $a^{\mathrm{T}} \rightarrow$(2 5 8 11 14 17 20 23 26 29 32 35 38 41 44 47 50 53 56 59 62 65 68 71 74 77 80 83 86 89 92 95 98 101 104 107)

   is a classic ascending arithmetic progression with $a_1 = 2$ and $r = 3$;

2. Let $L := 35 \ \ k := 1..L \ \ r_k := k$, then

   $r^{\mathrm{T}} \rightarrow$(1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35),

   if $a_1 := 2$ and $a_{k+1} := a_k + r_k$, then result

   $a^{\mathrm{T}} \rightarrow$(2 3 5 8 12 17 23 30 38 47 57 68 80 93 107 122 138 155 173 192 212 233 255 278 302 327 353 380 408 437 467 498 530 563 597 632);

   then we have a generalized ascending arithmetic progression but it is not a classical ascending arithmetic progression.

3. Let $L := 30$ $k := 1..L$ $r_k := k^2$, then

   $r^{\mathrm{T}} \rightarrow (1\ 4\ 9\ 16\ 25\ 36\ 49\ 64\ 81\ 100\ 121\ 144\ 169\ 196\ 225\ 256\ 289\ 324\ 361\ 400\ 441\ 484\ 529\ 576\ 625\ 676\ 729\ 784\ 841\ 900)$,

   if $a_1 := 2$ and $a_{k+1} := a_k + r_k$, then result

   $a^{\mathrm{T}} \rightarrow (2\ 3\ 7\ 16\ 32\ 57\ 93\ 142\ 206\ 287\ 387\ 508\ 652\ 821\ 1017\ 1242\ 1498\ 1787\ 2111\ 2472\ 2872\ 3313\ 3797\ 4326\ 4902\ 5527\ 6203\ 6932\ 7716\ 8557\ 9457)$;

   then we have a generalized ascending arithmetic progression but it is not a classical arithmetic progression.

4. Let $L := 25$ $k := 1..L$ $r_k := k^3$, then

   $r^{\mathrm{T}} \rightarrow (1\ 8\ 27\ 64\ 125\ 216\ 343\ 512\ 729\ 1000\ 1331\ 1728\ 2197\ 2744\ 3375\ 4096\ 4913\ 5832\ 6859\ 8000\ 9261\ 10648\ 12167\ 13824\ 15625)$,

   if $a_1 := 2$ and $a_{k+1} := a_k + r_k$, then result

   $a^{\mathrm{T}} \rightarrow (2\ 3\ 11\ 38\ 102\ 227\ 443\ 786\ 1298\ 2027\ 3027\ 4358\ 6086\ 8283\ 11027\ 14402\ 18498\ 23411\ 29243\ 36102\ 44102\ 53363\ 64011\ 76178\ 90002\ 105627)$;

   then we have a generalized ascending arithmetic progression but it is not a classical arithmetic progression.

5. Let $L := 38$ $k := 1..L$ $r_k := 1 + \mod(k-1, 6)$, then

   $r^{\mathrm{T}} \rightarrow (1\ 2\ 3\ 4\ 5\ 6\ 1\ 2\ 3\ 4\ 5\ 6\ 1\ 2\ 3\ 4\ 5\ 6\ 1\ 2\ 3\ 4\ 5\ 6\ 1\ 2\ 3\ 4\ 5\ 6\ 1\ 2\ 3\ 4\ 5\ 6\ 1\ 2)$,

   if $a_1 := 2$ and $a_{k+1} := a_k + r_k$, then result

   $a^{\mathrm{T}} \rightarrow (2\ 3\ 5\ 8\ 12\ 17\ 23\ 24\ 26\ 29\ 33\ 38\ 44\ 45\ 47\ 50\ 54\ 59\ 65\ 66\ 68\ 71\ 75\ 80\ 86\ 87\ 89\ 92\ 96\ 101\ 107\ 108\ 110\ 113\ 117\ 122\ 128\ 129\ 131)$;

   then we have a generalized ascending arithmetic progression but it is not a classical arithmetic progression.

6. If

   $r := (2\ 3\ 5\ 7\ 11\ 13\ 17\ 19\ 23\ 29\ 31\ 37\ 41\ 43\ 47\ 53\ 59\ 61\ 67\ 71\ 73\ 79\ 83\ 89\ 97\ 101\ 103\ 107\ 109\ 113\ 127\ 131\ 137\ 139\ 149)^{\mathrm{T}}$

i.e. sequence of rations is sequence of prime numbers, and $a_1 := 2$ $L := 35$ $k := 1..L$ $a_{k+1} := a_k + r_k$, then result

$a^{\mathrm{T}} \rightarrow$(2 4 7 12 19 30 43 60 79 102 131 162 199 240 283 330 383 442 503 570 641 714 793 876 965 1062 1163 1266 1373 1482 1595 1722 1853 1990 2129 2278);

then we have a generalized ascending arithmetic progression but it is not a classical arithmetic progression.

7. Let $L := 35$ $k := 1..L$ $r_k := -3$, then

   $r^{\mathrm{T}} \rightarrow$(–3 –3 –3 –3 –3 –3 –3 –3 –3 –3 –3 –3 –3 –3 –3 –3 –3 –3 –3 –3 –3 –3 –3 –3 –3 –3 –3 –3 –3 –3 –3 –3 –3 –3 –3),

   if $a_1 := 150$ and $a_{k+1} := a_k + r_k$, then result

   $a^{\mathrm{T}} \rightarrow$(150 147 144 141 138 135 132 129 126 123 120 117 114 111 108 105 102 99 96 93 90 87 84 81 78 75 72 69 66 63 60 57 54 51 48 45)

   is a classic descending arithmetic progression with $a_1 = 150$ and $r = -3$;

8. Let $L := 30$ $k := 1..L$ $r_k := -k$, then

   $r^{\mathrm{T}} \rightarrow$(–1 –2 –3 –4 –5 –6 –7 –8 –9 –10 –11 –12 –13 –14 –15 –16 –17 –18 –19 –20 –21 –22 –23 –24 –25 –26 –27 –28 –29 –30),

   if $a_1 := 500$ and $a_{k+1} := a_k + r_k$, then result

   $a^{\mathrm{T}} \rightarrow$(500 499 497 494 490 485 479 472 464 455 445 434 422 409 395 380 364 347 329 310 290 269 247 224 200 175 149 122 94 65 35);

   then we have a generalized descending arithmetic progression but it is not a classical descending arithmetic progression.

9. Let $L := 25$ $k := 1..L$ $r_k := -k^2$, then

   $r^{\mathrm{T}} \rightarrow$(–1 –4 –9 –16 –25 –36 –49 –64 –81 –100 –121 –144 –169 –196 –225 –256 –289 –324 –361 –400 –441 –484 –529 –576 -625),

   if $a_1 := 10000$ and $a_{k+1} := a_k + r_k$, then result

   $a^{\mathrm{T}} \rightarrow$(10000 9999 9995 9986 9970 9945 9909 9860 9796 9715 9615 9494 9350 9181 8985 8760 8504 8215 7891 7530 7130 6689 6205 5676 5100 4475);

then we have a generalized descending arithmetic progression but it is not a classical descending arithmetic progression.

10. Let $L := 20$ $k := 1..L$ $r_k := -k^3$, then

$r^{\mathrm{T}} \to (-1\ -8\ -27\ -64\ -125\ -216\ -343\ -512\ -729\ -1000\ -1331\ -1728\ -2197\ -2744\ -3375\ -4096\ -4913\ -5832\ -6859\ -8000)$,

if $a_1 := 50000$ and $a_{k+1} := a_k + r_k$, then result

$a^{\mathrm{T}} \to (50000\ 49999\ 49991\ 49964\ 49900\ 49775\ 49559\ 49216\ 48704\ 47975\ 46975\ 45644\ 43916\ 41719\ 38975\ 35600\ 31504\ 26591\ 20759\ 13900\ 5900)$;

then we have a generalized descending arithmetic progression but it is not a classical descending arithmetic progression.

11. Let $L := 30$ $k := 1..L$ $r_k := -(1 + \mod(k-1, 6))$, then

$r^{\mathrm{T}} \to (-1\ -2\ -3\ -4\ -5\ -6\ -1\ -2\ -3\ -4\ -5\ -6\ -1\ -2\ -3\ -4\ -5\ -6\ -1\ -2\ -3\ -4\ -5\ -6\ -1\ -2\ -3\ -4\ -5\ -6)$,

if $a_1 := 200$ and $a_{k+1} := a_k + r_k$, then result

$a^{\mathrm{T}} \to (200\ 199\ 197\ 194\ 190\ 185\ 179\ 178\ 176\ 173\ 169\ 164\ 158\ 157\ 155\ 152\ 148\ 143\ 137\ 136\ 134\ 131\ 127\ 122\ 116\ 115\ 113\ 110\ 106\ 101\ 95)$.

then we have a generalized descending arithmetic progression but it is not a classical descending arithmetic progression.

3.22 Non–Arithmetic Progression

Smarandache [2006] defines the series of numbers which are not arithmetic progression as the series of numbers that we have for all third term the relationship $a_{k+2} \neq a_{k+1} + r$, where $r = a_{k+1} - a_k$. This statement is equivalent to that all rations third term of the sequence is different from the other two previous terms.

We suggest new definitions for non-arithmetic progression. If we have a series of real numbers $\{a_k\}$, $k = 1, 2, \ldots$ we say that the series $r_k = a_{k+1} - a_k$, $k = 1, 2, \ldots$ is the series of ratios' series $\{a_k\}$.

Definition 3.56. The real series $\{a_k\}$, $k = 1,2,\ldots$is a non–arithmetic progression generalization if the ratios' series is a series of numbers who do not have the same sign.

Definition 3.57. The real series $\{a_k\}$, $k = 1,2,\ldots$ is a non–arithmetic progression classical if the ratios' series is a series of non–constant numbers.

Observation 3.58. Definition 3.57 includes the Smarandache [2006] definition that we have non–arithmetic progression if all the third term of the series does not verify the equality $a_{k+2} = a_{k+1} + r = 2a_{k+1} - a_k$ for any $k = 1,2,\ldots$.

It is obvious that any classical arithmetic progression is also an arithmetic progression generalization. Therefore, any series of non–arithmetic progression generalization is also a classical non–arithmetic progression..

Program 3.59. test program if is arithmetic progression. We consider the following assignments of texts.

t_1 :="*Classical increasing arithmetic progression*";

t_2 :="*Classical decreasing arithmetic progression*";

t_3 :="*Generalized increasing arithmetic progression but not classical*";

t_4 :="*Generalized decreasing arithmetic progression but not classical*";

t_5 :="*Non–generalized arithmetic progression*";

```
PVap(a) := | for k ∈ 1..last(a) − 1
           |   | r_k ← a_{k+1} − a_k
           | for k ∈ 2..last(r)
           |   | npac ← npac + 1  if r_k ≠ r_1
           |   | npag ← npag + 1  if sign(r_1 · r_k) = −1
           | if npac=0
           |   | return t_1 if sign(r_1) > 0
           |   | return t_2 if sign(r_1) < 0
           |   | return "Error." if r_1=0
           | if npac ≠ 0 ∧ npag=0
           |   | return t_3 if sign(r_1) > 0
           |   | return t_4 if sign(r_1) < 0
           |   | return "Error." if r_1=0
           | return t_5 if npag ≠ 0
```

Examples (for tracking easier the examples we considered only rows of integers):

1. Let $L := 100$ $k := 2..L$ $r1_k :=$ mod $(k-1, 2)$, then

   $r1^{\mathrm{T}} \rightarrow$(1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2)

   if $u1_1 := 1$ and $u1_k := u1_{k-1} + r1_{k-1}$, then result

   $u1^{\mathrm{T}} \rightarrow$(1 2 4 5 7 8 10 11 13 14 16 17 19 20 22 23 25 26 28 29 31 32 34 35 37 38 40 41 43 44 46 47 49 50 52 53 55 56 58 59 61 62 64 65 67 68 70 71 73 74 76 77 79 80 82 83 85 86 88 89 91 92 94 95 97 98 100 101 103 104 106 107 109 110 112 113 115 116 118 119 121 122 124 125 127 128 130 131 133 134 136 137 139 140 142 143 145 146 148 149)

   and $PVap(u1)$ ="*Generalized increasing arithmetic progression but not classical*".

2. Let $L := 100$ $k := 2..L$ $r2_k :=$ mod $(k-1, 3)$, then

   $r2^{\mathrm{T}} \rightarrow$(1 2 3 1)

   if $u2_1 := 1$ and $u2_k := u2_{k-1} + r2_{k-1}$, then result

   $u2^{\mathrm{T}} \rightarrow$(1 2 4 7 8 10 13 14 16 19 20 22 25 26 28 31 32 34 37 38 40 43 44 46 49 50 52 55 56 58 61 62 64 67 68 70 73 74 76 79 80 82 85 86 88 91 92 94 97 98 100 103 104 106 109 110 112 115 116 118 121 122 124 127 128 130 133 134 136 139 140 142 145 146 148 151 152 154 157 158 160 163 164 166 169 170 172 175 176 178 181 182 184 187 188 190 193 194 196 199)

   and $PVap(u2)$ ="*Generalized increasing arithmetic progression but not classical*".

3. Let $L := 100$ $k := 2..L$ $r3_k :=$ mod $(k-1, 4)$, then

   $r3^{\mathrm{T}} \rightarrow$(1 2 3 4 1 2 3 4)

if $u3_1 := 1$ and $u3_k := u3_{k-1} + r3_{k-1}$, then result

$u4^{\mathrm{T}} \rightarrow$(1 2 4 7 11 12 14 17 21 22 24 27 31 32 34 37 41 42 44 47 51 52 54 57 61 62 64 67 71 72 74 77 81 82 84 87 91 92 94 97 101 102 104 107 111 112 114 117 121 122 124 127 131 132 134 137 141 142 144 147 151 152 154 157 161 162 164 167 171 172 174 177 181 182 184 187 191 192 194 197 201 202 204 207 211 212 214 217 221 222 224 227 231 232 234 237 241 242 244 247)

and $PVap(u3)$ ="*Generalized increasing arithmetic progression but not classical*".

4. Let $L := 100$ $k := 2..L$ $r4_k :=$ mod $(k-1,5)$, then

   $r4^{\mathrm{T}} \rightarrow$(1 2 3 4 5 1 2 3 4 5 1 2 3 4 5 1 2 3 4 5 1 2 3 4 5 1 2 3 4 5 1 2 3 4 5 1 2 3 4 5 1 2 3 4 5 1 2 3 4 5 1 2 3 4 5 1 2 3 4 5 1 2 3 4 5 1 2 3 4 5 1 2 3 4 5 1 2 3 4 5 1 2 3 4 5 1 2 3 4 5 1 2 3 4 5 1 2 3 4 5)

   if $u4_1 := 1$ and $u4_k := u4_{k-1} + r4_{k-1}$, then result

   $u4^{\mathrm{T}} \rightarrow$(1 2 4 7 11 16 17 19 22 26 31 32 34 37 41 46 47 49 52 56 61 62 64 67 71 76 77 79 82 86 91 92 94 97 101 106 107 109 112 116 121 122 124 127 131 136 137 139 142 146 151 152 154 157 161 166 167 169 172 176 181 182 184 187 191 196 197 199 202 206 211 212 214 217 221 226 227 229 232 236 241 242 244 247 251 256 257 259 262 266 271 272 274 277 281 286 287 289 292 296)

   and $PVap(u4)$ ="*Generalized increasing arithmetic progression but not classical*".

Examples non-arithmetic progression generalization:

- Let $L := 100$ $k := 2..L$ $r5_k := floor(-1 + rnd(10))$, then

  $r5^{\mathrm{T}} \rightarrow$(1 2 0 –1 6 5 5 –1 –1 3 6 5 2 –1 0 2 1 3 3 5 3 7 5 –1 7 2 –1 2 –1 7 3 5 7 4 6 4 4 5 –1 2 6 7 6 7 4 6 0 6 7 5 5 1 6 6 –1 7 2 3 0 8 5 0 1 7 2 1 1 8 –1 0 –1 5 3 –1 5 5 4 –1 1 7 –1 6 7 1 5 7 7 8 3 2 3 –1 6 –1 5 0 –1 –1 5 8)

  if $u5_1 := 10$ and $u5_k := u5_{k-1} + r5_{k-1}$, then result

$u5^{\mathrm{T}} \rightarrow$(10 11 13 13 12 18 23 28 27 26 29 35 40 42 41 41 43 44 47 50 55 58 65 70 69 76 78 77 79 78 85 88 93 100 104 110 114 118 123 122 124 130 137 143 150 154 160 160 166 173 178 183 184 190 196 195 202 204 207 207 215 220 220 221 228 230 231 232 240 239 239 238 243 246 245 250 255 259 258 259 266 265 271 278 279 284 291 298 306 309 311 314 313 319 318 323 323 322 321 326)

and $PVap(u5) =$"*Non–generalized arithmetic progression*".

- Let $L := 100$ $k := 2..L$ $r6_k := floor(2\sin(k)^2 + 3\cos(k)^3 + 5\sin(k)^2\cos(k)^3)$, then

$r6^{\mathrm{T}} \rightarrow$(1 1 –3 –1 2 3 3 1 –3 –3 1 3 3 1 –2 –3 1 2 3 2 0 –3 0 2 3 2 1 –3 –2 1 3 3 1 –3 –3 1 3 3 2 –1 –3 1 2 3 2 1 –3 –1 2 3 3 1 –3 –2 1 3 3 1 –2 –3 1 2 3 2 0 –3 0 2 3 2 1 –3 –2 1 3 3 1 –3 –3 1 3 3 1 –1 –3 1 2 3 2 0 –3 –1 2 3 3 1 –3 –2 1 3)

if $u6_1 := 100$ and $u6_k := u6_{k-1} + r6_{k-1}$, then result

$u6^{\mathrm{T}} \rightarrow$(100 101 102 99 98 100 103 106 107 104 101 102 105 108 109 107 104 105 107 110 112 112 109 109 111 114 116 117 114 112 113 116 119 120 117 114 115 118 121 123 122 119 120 122 125 127 128 125 124 126 129 132 133 130 128 129 132 135 136 134 131 132 134 137 139 139 136 136 138 141 143 144 141 139 140 143 146 147 144 141 142 145 148 149 148 145 146 148 151 153 153 150 149 151 154 157 158 155 153 154)

and *PV(u6)* ="*Non–generalized arithmetic progression*"

3.23 Generalized Geometric Progression

A classical generalized geometric progression is defined by: $g_1 \in \mathbb{R}$, the first term of progression, $\rho \neq 0$, $\rho \in \mathbb{R}$ progression ratio (if $\rho > 1$ then we have an ascending progression, if $0 < \rho < 1$ then we have a descending progression) and the formula $g_{k+1} = g_k \cdot \rho = g_1 \cdot \rho^k$, for any $k \in \mathbb{N}^*$, where g_{k+1} the term of rank $k+1$.

Obviously we can consider ascending progression of integers (or natural numbers) or descending progression of integers (or natural numbers), where $g_1 \in \mathbb{Z}$, (or $g_1 \in \mathbb{N}$), $\rho \in \mathbb{Z}^*$ (or $\rho \in \mathbb{N}$) and $g_{k+1} = g_k \cdot \rho = g_1 \cdot \rho^k$.

Consider the following generalization of geometric progressions. Let $g_1 \in \mathbb{N}$ the first term of the geometric progression and ρ_k a series of positive real

supraunitary numbers in ascending progressions or a series of real subunitary numbers in descending progression. We call the series $\{\rho_k\}$ il the series of generalized geometric progression ratios. The term g_{k+1} is defined by formula

$$g_{k+1} = g_k \cdot \rho_k = g_1 \cdot \prod_{j=1}^{k} \rho_j \ ,$$

for any $k \in \mathbb{N}^*$.

Examples (we preferred to give examples of progression of integers for easier browsing text):

1. Let $L := 20$ $k := 1..L$ $\rho_k := 3$, then

   $\rho^{\mathrm{T}} \to (3\ 3\ 3\ 3\ 3\ 3\ 3\ 3\ 3\ 3\ 3\ 3\ 3\ 3\ 3\ 3\ 3\ 3\ 3\ 3)$

   if $g_1 := 2$ and $g_{k+1} := g_k \cdot \rho_k$, then result that

   $g^{\mathrm{T}} \to$(2 6 18 54 162 486 1458 4374 13122 39366 118098 354294 1062882 3188646 9565938 28697814 86093442 258280326 774840978 2324522934 6973568802)

   which is a classical ascending geometric progression with $g_1 = 2$ and $\rho = 3$;

2. Let $L := 15$ $k := 1..L$ $\rho_k := k$, then

   $\rho^{\mathrm{T}} \to (2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12\ 13\ 14\ 15\ 16)$

   if $g_1 := 1$ and $g_{k+1} := g_k \cdot \rho_k$, then result that

   $g^{\mathrm{T}} \to$(1 2 6 24 120 720 5040 40320 362880 3628800 39916800 479001600 6227020800 87178291200 1307674368000 20922789888000)

   which is factorial sequence that is a generalized ascending geometric progression but is not classical geometric progression;

3. Let $L := 13$ $k := 1..L$ $\rho_k := (k+1)^2$, then

   $\rho^{\mathrm{T}} \to (4\ 9\ 16\ 25\ 36\ 49\ 64\ 81\ 100\ 121\ 144\ 169\ 196)$

   if $g_1 := 1$ and $g_{k+1} := g_k \cdot \rho_k$, then we obtain sequence

   $g^{\mathrm{T}} \to$(1 4 36 576 14400 518400 25401600 1625702400 131681894400 13168189440000 1593350922240000 229442532802560000 38775788043632640000 7600054456551997440000);

which is a generalized ascending geometric progression but is not classical geometric progression.

4. Let $L := 10 \ \ k := 1..L \ \ \rho_k := (k+1)^3$, then

$$\rho^{\mathrm{T}} \rightarrow (8 \ \ 27 \ \ 64 \ \ 125 \ \ 216 \ \ 343 \ \ 512 \ \ 729 \ \ 1000 \ \ 1331)$$

if $g_1 := 7$ and $g_{k+1} := g_k \cdot \rho_k$, then result that

$$g^{\mathrm{T}} \rightarrow (7 \ \ 56 \ \ 1512 \ \ 96768 \ \ 12096000 \ \ 2612736000 \ \ 896168448000 \ \ 458838245376000 \ \ 334493080879104000 \ \ 334493080879104000000 \ \ 445210290650087424000000);$$

which is a generalized ascending geometric progression but is not classical geometric progression.

5. Let $L := 15 \ \ k := 1..L \ \ \rho_k := 3 + \mod(k-1, 6)$, then

$$\rho^{\mathrm{T}} \rightarrow (3 \ \ 4 \ \ 5 \ \ 6 \ \ 7 \ \ 8 \ \ 3 \ \ 4 \ \ 5 \ \ 6 \ \ 7 \ \ 8 \ \ 3 \ \ 4 \ \ 5)$$

if $g_1 := 11$ and $g_{k+1} := g_k \cdot \rho_k$, then we obtain

$$g^{\mathrm{T}} \rightarrow (11 \ \ 33 \ \ 132 \ \ 660 \ \ 3960 \ \ 27720 \ \ 221760 \ \ 665280 \ \ 2661120 \ \ 13305600 \ \ 79833600 \ \ 558835200 \ \ 4470681600 \ \ 13412044800 \ \ 53648179200 \ \ 268240896000);$$

which is a generalized ascending geometric progression but is not classical geometric progression.

6. Let $L := 10 \ \ k := 1..L \ \ \rho_k := \frac{1}{3}$, then

$$\rho^{\mathrm{T}} \rightarrow \left(\frac{1}{3} \ \ \frac{1}{3} \ \ \frac{1}{3} \ \ \frac{1}{3} \ \ \frac{1}{3} \ \ \frac{1}{3} \ \ \frac{1}{3} \ \ \frac{1}{3} \ \ \frac{1}{3} \ \ \frac{1}{3}\right)$$

if $g_1 := 3^{10}$ and $g_{k+1} := g_k \cdot \rho_k$, then result that

$$g^{\mathrm{T}} \rightarrow (59049 \ \ 19683 \ \ 6561 \ \ 2187 \ \ 729 \ \ 243 \ \ 81 \ \ 27 \ \ 9 \ \ 3 \ \ 1)$$

which is a classic descending geometric progression with $g_1 = 59049$ and $\rho = \frac{1}{3}$;

7. Let $L := 10 \ \ k := 1..L \ \ \rho_k := 2^{-k}$, then

$$\rho^{\mathrm{T}} \rightarrow \left(\frac{1}{2} \ \ \frac{1}{4} \ \ \frac{1}{8} \ \ \frac{1}{16} \ \ \frac{1}{32} \ \ \frac{1}{64} \ \ \frac{1}{128} \ \ \frac{1}{256} \ \ \frac{1}{512} \ \ \frac{1}{1024}\right)$$

if $g_1 := 2^{55}$ and $g_{k+1} := g_k \cdot \rho_k$, then result that

$g^{\mathrm{T}} \rightarrow$(36028797018963968 18014398509481984 4503599627370496 562949953421312 35184372088832 1099511627776 17179869184 134217728 524288 1024 1);

which is a generalized descending geometric progression but is not classical geometric progression.

8. Let $L := 15$ $k := 1..L$ $\rho_k := 2^{-[1+ \mod(k-1,6)]}$, then

$$\rho^{\mathrm{T}} \rightarrow \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \frac{1}{32} & \frac{1}{64} & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \frac{1}{32} & \frac{1}{64} & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} \end{pmatrix}$$

if $g_1 := 2^{48}$ and $g_{k+1} := g_k \cdot \rho_k$, then result that

$g^{\mathrm{T}} \rightarrow$(281474976710656 140737488355328 35184372088832 4398046511104 274877906944 8589934592 134217728 67108864 16777216 2097152 131072 4096 64 32 8 1);

which is a generalized descending geometric progression but is not classical geometric progression.

3.24 Non–Geometric Progression

If we have a series of real numbers $\{g_k\}$, $k = 1, 2, \ldots$ we say that the series $\rho_k = \frac{a_{k+1}}{a_k}$, $k = 1, 2, \ldots$ is the series of ratios' series $\{g_k\}$.

Definition 3.60. The series $\{g_k\}$, $k = 1, 2, \ldots$ is a non–generalized geometric progression if ratios' series $\{\rho_k\}$ is a series of number not supraunitary (or subunitary).

Definition 3.61. The series $\{g_k\}$, $k = 1, 2, \ldots$ is a non–classical geometric progression if ratios' series $\{\rho_k\}$ is a series of inconstant numbers.

Observation 3.62. It is obvious that any classical geometric progression is a generalized geometric progression. Therefore, any series of non-generalized geometric progression is also non–classical geometric progression.

Program 3.63. for geometric progression testing. We consider the following texts' assignments.

$t_1 :=$"*Classical increasing geometric progression*";

$t_2 :=$"*Classical decreasing geometric progression*";

$t_3 :=$"*Generalized increasing geometric progression but not classical*";

$t_4 :=$"*Generalized decreasing geometric progression but not classical*";

$t_5 :=$"*Non–generalized geometric progression*";

$$
PVgp(g) := \left|
\begin{array}{l}
for\ \ k \in 1..last(g)-1 \\
\quad \left| \ \rho_k \leftarrow \dfrac{g_{k+1}}{g_k} \right. \\
for\ \ k \in 2..last(q) \\
\quad \left| \begin{array}{l} npac \leftarrow npac+1\ \ if\ \rho_k \neq \rho_1 \\ npag \leftarrow npag+1\ \ if\ \neg[(\rho_1 > 1 \wedge \rho_k > 1) \vee (\rho_1 < 1 \wedge \rho_k < 1)] \end{array} \right. \\
if\ \ npac = 0 \\
\quad \left| \begin{array}{l} return\ t_1\ \ if\ \ \rho_1 > 1 \\ return\ t_2\ \ if\ \ 0 < \rho_1 < 1 \\ return\ \text{"Error."}\ \ if\ \ \rho_1 \leq 0 \vee \rho_1 = 1 \end{array} \right. \\
if\ \ npac \neq 0 \wedge npag = 0 \\
\quad \left| \begin{array}{l} return\ t_3\ \ if\ \ \rho_1 > 1 \\ return\ t_4\ \ if\ \ 0 < \rho_1 < 1 \\ return\ \text{"Error."}\ \ if\ \rho_1 \leq 0 \vee \rho_1 = 1 \end{array} \right. \\
return\ t_5\ \ if\ \ npag \neq 0
\end{array}
\right.
$$

Examples:

1. Let $\rho 1_1 := \frac{6}{5}$ $L := 10$ $k := 2..L$ $\rho 1_k := \frac{6}{5}$, then

   $$\rho 1^{\mathrm{T}} \rightarrow \begin{pmatrix} \frac{6}{5} & \frac{6}{5} & \frac{6}{5} & \frac{6}{5} & \frac{6}{5} & \frac{6}{5} & \frac{6}{5} & \frac{6}{5} & \frac{6}{5} & \frac{6}{5} \end{pmatrix}$$

   if $w1_1 := 1$ and $w1_k := w1_{k-1} \cdot \rho 1_{k-1}$, then result

   $$w1^{\mathrm{T}} \rightarrow \Big(1\ \ \frac{6}{5}\ \ \frac{36}{25}\ \ \frac{216}{125}\ \ \frac{1296}{625}\ \ \frac{7776}{3125}\ \ \frac{46656}{15625}\ \ \frac{279936}{78125}\ \ \frac{1679616}{390625}$$
   $$\frac{10077696}{1953125}\Big)$$

   and $PVgp(w1) =$"*Classical increasing geometric progression*".

2. Let $\rho 2_1 := \frac{9}{10}$ $L := 10$ $k := 2..L$ $\rho 2_k := \frac{9}{10}$, then

   $$\rho 2^{\mathrm{T}} \rightarrow \begin{pmatrix} \frac{9}{10} & \frac{9}{10} & \frac{9}{10} & \frac{9}{10} & \frac{9}{10} & \frac{9}{10} & \frac{9}{10} & \frac{9}{10} & \frac{9}{10} & \frac{9}{10} \end{pmatrix}$$

   if $w2_1 := 10^9$ and $w2_k := w2_{k-1} \cdot \rho 2_{k-1}$, then result

   $$w2^{\mathrm{T}} \rightarrow (1000000000\ \ 900000000\ \ 810000000\ \ 729000000\ \ 656100000$$
   $$590490000\ \ 531441000\ \ 478296900\ \ 430467210\ \ 387420489)$$

and $PVgp(w2)$ ="*Classical decreasing geometric progression*".

3. Let $\rho3_1 := 2\ L := 17\ \ k := 2..L\ \ \rho3_k := q3_{k-1} + 1$, then

   $$\rho3^{\mathrm{T}} \to (2\ \ 3\ \ 4\ \ 5\ \ 6\ \ 7\ \ 8\ \ 9\ \ 10\ \ 11\ \ 12\ \ 13\ \ 14\ \ 15\ \ 16\ \ 17\ \ 18)$$

   if $w3_1 := 1$ and $w3_k := w3_{k-1} \cdot \rho3_{k-1}$, then result

   $$w3^{\mathrm{T}} \to (1\quad 2\quad 6\quad 24\quad 120\quad 720\quad 5040\quad 40320\quad 362880\quad 3628800\quad 39916800\quad 479001600\quad 6227020800\quad 87178291200\quad 1307674368000\quad 20922789888000\quad 355687428096000)$$

   and $PVgp(w3)$ ="*Generalized increasing geometric progression but not classical*". Note that the series $w3$ is the series of factorials up to 17!.

4. Let $\rho4_1 := \frac{1}{2}\ L := 10\ \ k := 2..L\ \ \rho4_k := \rho4_{k-1} \cdot \frac{1}{2}$, then

   $$\rho4^{\mathrm{T}} \to \left(\frac{1}{2}\ \ \frac{1}{4}\ \ \frac{1}{8}\ \ \frac{1}{16}\ \ \frac{1}{32}\ \ \frac{1}{64}\ \ \frac{1}{128}\ \ \frac{1}{256}\ \ \frac{1}{512}\ \ \frac{1}{1024}\right)$$

   if $w4_1 := 1$ and $w4_k := w4_{k-1} \cdot \rho4_{k-1}$, then result

   $$w4^{\mathrm{T}} \to \left(1\ \ \frac{1}{2}\ \ \frac{1}{8}\ \ \frac{1}{64}\ \ \frac{1}{1024}\ \ \frac{1}{32768}\ \ \frac{1}{2097152}\ \ \frac{1}{268435456}\ \ \frac{1}{68719476736}\ \ \frac{1}{35184372088832}\right)$$

   and $PVgp(w4)$ ="*Generalized decreasing geometric progression but not classical*".

5. Let $\rho5_1 := \frac{1}{2}\ L := 10\ \ k := 2..L\ \ \rho5_k := \rho4_{k-1} \cdot 3^{(-1)^k}$, then

   $$\rho5^{\mathrm{T}} \to \left(\frac{1}{2}\ \ \frac{3}{2}\ \ \frac{1}{2}\ \ \frac{3}{2}\ \ \frac{1}{2}\ \ \frac{3}{2}\ \ \frac{1}{2}\ \ \frac{3}{2}\ \ \frac{1}{2}\ \ \frac{3}{2}\right)$$

   if $w5_1 := 1$ and $w5_k := w5_{k-1} \cdot \rho5_{k-1}$, then result

   $$w5^{\mathrm{T}} \to \left(1\ \ \frac{1}{2}\ \ \frac{3}{4}\ \ \frac{3}{8}\ \ \frac{9}{16}\ \ \frac{9}{32}\ \ \frac{27}{64}\ \ \frac{27}{128}\ \ \frac{81}{256}\ \ \frac{81}{512}\right)$$

   and $PVgp(w5)$ ="*Non-generalized geometric progression*".

Chapter 4

Special numbers

4.1 Numeration Bases

4.1.1 Prime Base

We defined over the set of natural numbers the following infinite base: $p_0 = 1$, and for $k \in \mathbb{N}^*$ $p_k = prime_k$ is the k-th prime number. We proved that every positive integer $a \in \mathbb{N}^*$ may be uniquely written in the prime base as:

$$a = \overline{a_m \dots a_1 a_0}_{(pb)} = \sum_{k=0}^{m} a_k p_k \ ,$$

where $a_k = 0$ or 1 for $k = 0, 1, \dots, m-1$ and of course $a_m = 1$, in the following way:

- if $p_m \le a < p_{m+1}$ then $a = p_m + r_1$;
- if $p_k \le r_1 < p_{k+1}$ then $r_1 = p_k + r_2$, $k < m$;
- and so on until one obtains a rest $r_j = 0$.

Therefore, any number may be written as a sum of prime numbers $+e$, where $e = 0$ or 1. Thus we have

$$\begin{aligned}
2_{(10)} &= 1\cdot 2 + 0\cdot 1 = 10_{(pb)} \ , \\
3_{(10)} &= 1\cdot 3 + 0\cdot 2 + 0\cdot 1 = 100_{(pb)} \ , \\
4_{(10)} &= 1\cdot 3 + 0\cdot 2 + 1\cdot 1 = 101_{(pb)} \ , \\
5_{(10)} &= 1\cdot 5 + 0\cdot 3 + 0\cdot 2 + 0\cdot 1 = 1000_{(pb)} \ ,
\end{aligned}$$

$$\begin{aligned}
6_{(10)} &= 1\cdot5+0\cdot3+0\cdot2+1\cdot1 = 1001_{(pb)}\ ,\\
7_{(10)} &= 1\cdot7+0\cdot5+0\cdot3+0\cdot2+0\cdot1 = 10000_{(pb)}\ ,\\
8_{(10)} &= 1\cdot7+0\cdot5+0\cdot3+0\cdot2+1\cdot1 = 10001_{(pb)}\ ,\\
9_{(10)} &= 1\cdot7+0\cdot5+0\cdot3+1\cdot2+0\cdot1 = 10010_{(pb)}\ ,\\
10_{(10)} &= 1\cdot7+0\cdot5+1\cdot3+0\cdot2+0\cdot1 = 10100_{(pb)}\ .
\end{aligned}$$

If we use the *ipp* function, given by 2.40, then a is written in the prime base as:

$$a = ipp(a) + ipp(a - ipp(a)) + ipp(a - ipp(a) - ipp(a - ipp(a))) + \dots\ ,$$

or

$$a = ipp(a) + ipp(ppi(a)) + ipp(ppi(a) - ipp(ppi(a))) + \dots$$

where the function *ppi* given by 2.46.

Example 4.1. Let $a = 35$, then

$$\begin{aligned}
&ipp(35) + ipp(35 - ipp(35)) + ipp(35 - ipp(35) - ipp(35 - ipp(35)))\\
&\qquad = 31 + 3 + 1 = 35
\end{aligned}$$

or

$$ipp(35) + ipp(ppi(35)) + ipp(ppi(35) - ipp(ppi(35))) = 31 + 3 + 1 = 35.$$

This base is important for partitions with primes.

Program 4.2. number generator based numeration of prime numbers, denoted (pb).

$$PB(n) := \left| \begin{array}{l}
return\ \ 1\ \ if\ \ n{=}1\\
v_{\pi(ipp(n))+1} \leftarrow 1\\
r \leftarrow ppi(n)\\
while\ r \neq 1 \wedge r \neq 0\\
\quad \left| \begin{array}{l} v_{\pi(ipp(r))+1} \leftarrow 1\\ r \leftarrow ppi(r) \end{array}\right.\\
v_1 \leftarrow 1\ \ if\ \ r{=}1\\
return\ \ reverse(v)^{\mathrm{T}}
\end{array}\right.$$

The program uses the programs: π of counting the prime numbers, 2.3, *ipp* inferior prime part 2.40, *ppi*, inferior prime complements, 2.46, utilitarian function Mathcad *reverse*.

Using the sequence $n := 1..25$, $v_n = PB(n)$ the vector v was generated, which contains the numbers from 1 to 25 written on the basis (pb):

Table 4.1: Numbers in base (pb)

$n_{(10)} = n_{(pb)}$
1=1
2=10
3=100
4=101
5=1000
6=1001
7=10000
8=10001
9=10010
10=10100
11=100000
12=100001
13=1000000
14=1000001
15=1000010
16=1000100
17=10000000
18=10000001
19=100000000
20=100000001
21=100000010
22=100000100
23=1000000000
24=1000000001
25=1000000010
26=1000000100
27=1000000101
28=1000001000
29=10000000000
30=10000000001
31=100000000000
32=100000000001
33=100000000010
34=100000000100
35=100000000101
36=100000001000
37=1000000000000

Continued on next page

$n_{(10)} = n_{(pb)}$
38=1000000000001
39=1000000000010
40=1000000000100
41=10000000000000
42=10000000000001
43=100000000000000
44=100000000000001
45=100000000000010
46=100000000000100
47=1000000000000000

4.1.2 Square Base

We defined over the set of natural numbers the following infinite base: for $k \in \mathbb{N}$, $s_k = k^2$, denoted (sb).

Each number $a \in \mathbb{N}$ can be written in the square base (sb). We proved that every positive integer a may be uniquely written in the square base as:

$$a = \overline{a_m \dots a_1 a_0}_{(sb)} = \sum_{k=0}^{m} a_k \cdot s_k \ ,$$

with $a_k = 0 \lor a_k = 1$ for $k \geq 2$, $a_1 \in \{0, 1, 2\}$, $a_0 \in \{0, 1, 2, 3\}$ and of course $a_m = 1$, in the following way:

- if $s_m \leq a < s_{m+1}$, then $a = s_m + r_1$;
- if $s_k \leq r_1 < s_{k+1}$, then $r_1 = s_k + r_2$, $k < m$ and so on until one obtains a rest $r_j = 0$, $j < m$.

Therefore, any number may be written as a sum of squares+e (1 not counted as a square – being obvious), where $e \in \{0, 1, 2, 3\}$. Examples: $4 = 2^2 + 0$, $5 = 2^2 + 1$, $6 = 2^2 + 2$, $7 = 2^2 + 3$, $8 = 2 \cdot 2^2 + 0$, $9 = 3^2 + 0$.

Program 4.3. for transforming a number written in base (10) based on the numeration (sb).

$$SB(n) := \left| \begin{array}{l} return\ \ 0\ \ if\ \ n{=}0 \\ v_{\sqrt{isp(n)}} \leftarrow 1 \\ r \leftarrow spi(n) \\ k \leftarrow \sqrt{isp(r)} \end{array} \right.$$

$$\left|\begin{array}{l} while\ r > 3 \\ \quad \left| \begin{array}{l} v_k \leftarrow v_k + 1 \\ r \leftarrow spi(r) \\ k \leftarrow \sqrt{isp(r)} \end{array}\right. \\ v_1 \leftarrow v_1 + r \\ return\ \ reverse(v)\cdot Vb(10, last(v)) \end{array}\right.$$

The program uses the following user functions: *isp* given by 2.48, *spi* given by 2.51, *Vb* which returns the vector $(b^m\ b^{m-1}\ \ldots b^0)^{\mathrm{T}}$ and the utilitarian function Mathcad *reverse*.

The numbers from 1 to 100 generated by the program *SB* are: 1, 2, 3, 10, 11, 12, 13, 20, 100, 101, 102, 103, 110, 111, 112, 1000, 1001, 1002, 1003, 1010, 1011, 1012, 1013, 1020, 10000, 10001, 10002, 10003, 10010, 10011, 10012, 10013, 10020, 10100, 10101, 100000, 100001, 100002, 100003, 100010, 100011, 100012, 100013, 100020, 100100, 100101, 100102, 100103, 1000000, 1000001, 1000002, 1000003, 1000010, 1000011, 1000012, 1000013, 1000020, 1000100, 1000101, 1000102, 1000103, 1000110, 1000111, 10000000, 10000001, 10000002, 10000003, 10000010, 10000011, 10000012, 10000013, 10000020, 10000100, 10000101, 10000102, 10000103, 10000110, 10000111, 10000112, 10001000, 100000000, 100000001, 100000002, 100000003, 100000010, 100000011, 100000012, 100000013, 100000020, 100000100, 100000101, 100000102, 100000103, 100000110, 100000111, 100000112, 100001000, 100001001, 100001002, 1000000000 .

4.1.3 Cubic Base

We defined over the set of natural numbers the following infinite base: for $k \in \mathbb{N}$, $c_k = k^3$, denoted (cb).

Each number $a \in \mathbb{N}$ can be written in the square base (cb). We proved that every positive integer a may be uniquely written in the cubic base as:

$$a = \overline{a_m \ldots a_1 a_0}_{(cb)} = \sum_{k=0}^{m} a_k \cdot c_k \ ,$$

with $a_k = 0 \vee a_k = 1$ for $k \geq 2$, $a_1 \in \{0,1,2\}$, $a_0 \in \{0,1,2,\ldots,7\}$ and of course $a_m = 1$, in the following way:

- if $c_m \leq a < c_{m+1}$, then $a = c_m + r_1$;
- if $c_k \leq r_1 < c_{k+1}$, then $r_1 = c_k + r_2$, $k < m$ and so on until one obtains a rest $r_j = 0$, $j < m$.

Therefore, any number may be written as a sum of cub+e (1 not counted as a square – being obvious), where $e \in \{0,1,2,\ldots,7\}$.

Examples: $9 = 2^3+1$, $10 = 2^3+1$, $11 = 2^3+2$, $12 = 2^3+3$, $13 = 2^3+4$, $14 = 2^3+5$, $15 = 2^3+6$, $16 = 2\cdot 2^3$, $17 = 2\cdot 2^3+1$, $18 = 2\cdot 2^3+2$, $19 = 2\cdot 2^3+3$, $20 = 2\cdot 2^3+4$, $21 = 2\cdot 2^3+5$, $22 = 2\cdot 2^3+6$, $23 = 2\cdot 2^3+7$, $24 = 3\cdot 2^3$, $25 = 3\cdot 2^3+1$, $26 = 3\cdot 2^3+2$, $27 = 3^3$.

Program 4.4. for transforming a number written in base (10) based on the numeration (*cb*).

$$
\begin{array}{l|l}
CB(n) := & return\ \ 0\ \ if\ \ n{=}0 \\
& k \leftarrow \sqrt[3]{icp(n)} \\
& v_k \leftarrow 1 \\
& r \leftarrow cpi(n) \\
& k \leftarrow \sqrt[3]{icp(r)} \\
& while\ r > 7 \\
& \quad \left| \begin{array}{l} v_k \leftarrow v_k + 1 \\ r \leftarrow cpi(r) \\ k \leftarrow \sqrt[3]{icp(r)} \end{array} \right. \\
& v_1 \leftarrow v_1 + r \\
& return\ \ reverse(v) \cdot Vb(10, last(v))
\end{array}
$$

The program uses the following user functions: *icp* given by 2.53, *cpi* given by 2.56, *Vb* which returns the vector $(b^m\ b^{m-1}\ \ldots b^0)^{\mathrm{T}}$ and the utilitarian function Mathcad *reverse*.

The natural numbers from 1 to 64 generated by the program *CB* are: 1, 2, 3, 4, 5, 6, 7, 10, 11, 12, 13, 14, 15, 16, 17, 20, 21, 22, 23, 24, 25, 26, 27, 30, 31, 32, 100, 101, 102, 103, 104, 105, 106, 107, 110, 111, 112, 113, 114, 115, 116, 117, 120, 121, 122, 123, 124, 125, 126, 127, 130, 131, 132, 200, 201, 202, 203, 204, 205, 206, 207, 210, 211, 1000 .

4.1.4 Factorial Base

We defined over the set of natural numbers the following infinite base: for $k \in \mathbb{N}^*$, $f_k = k!$, denoted (fb). We proved that every positive integer a may be uniquely written in the factorial base as:

$$a = \overline{a_m \ldots a_1 a_0}_{(fb)} = \sum_{k=0}^{m} a_k \cdot f_k\ ,$$

with all $a_k = 0, 1, \ldots, k$, for $k \in \mathbb{N}^*$, in the following way:

$$
\begin{aligned}
1_{(10)} &= 1\cdot 1! = 1_{(fb)}\\
2_{(10)} &= 1\cdot 2! + 0\cdot 1! = 10_{(fb)}\\
3_{(10)} &= 1\cdot 2! + 1\cdot 1! = 11_{(fb)}\\
4_{(10)} &= 2\cdot 2! + 0\cdot 1! = 20_{(fb)}\\
5_{(10)} &= 2\cdot 2! + 1\cdot 1! = 21_{(fb)}\\
6_{(10)} &= 1\cdot 3! + 0\cdot 2! + 0\cdot 1! = 100_{(fb)}\\
7_{(10)} &= 1\cdot 3! + 0\cdot 2! + 1\cdot 1! = 101_{(fb)}\\
8_{(10)} &= 1\cdot 3! + 1\cdot 2! + 0\cdot 1! = 110_{(fb)}\\
9_{(10)} &= 1\cdot 3! + 1\cdot 2! + 1\cdot 1! = 111_{(fb)}\\
10_{(10)} &= 1\cdot 3! + 2\cdot 2! + 0\cdot 1! = 120_{(fb)}\\
11_{(10)} &= 1\cdot 3! + 2\cdot 2! + 1\cdot 1! = 121_{(fb)}\\
12_{(10)} &= 2\cdot 3! + 0\cdot 2! + 0\cdot 1! = 200_{(fb)}\\
13_{(10)} &= 2\cdot 3! + 0\cdot 2! + 1\cdot 1! = 201_{(fb)}\\
14_{(10)} &= 2\cdot 3! + 1\cdot 2! + 0\cdot 1! = 210_{(fb)}\\
15_{(10)} &= 2\cdot 3! + 1\cdot 2! + 1\cdot 1! = 211_{(fb)}\\
16_{(10)} &= 2\cdot 3! + 2\cdot 2! + 0\cdot 1! = 220_{(fb)}\\
17_{(10)} &= 2\cdot 3! + 2\cdot 2! + 1\cdot 1! = 221_{(fb)}\\
18_{(10)} &= 3\cdot 3! + 0\cdot 2! + 0\cdot 1! = 300_{(fb)}\\
19_{(10)} &= 3\cdot 3! + 0\cdot 2! + 1\cdot 1! = 301_{(fb)}\\
20_{(10)} &= 3\cdot 3! + 1\cdot 2! + 0\cdot 1! = 301_{(fb)}\\
21_{(10)} &= 3\cdot 3! + 1\cdot 2! + 1\cdot 1! = 311_{(fb)}\\
22_{(10)} &= 3\cdot 3! + 2\cdot 2! + 0\cdot 1! = 320_{(fb)}\\
23_{(10)} &= 3\cdot 3! + 2\cdot 2! + 1\cdot 1! = 321_{(fb)}\\
24_{(10)} &= 1\cdot 4! + 0\cdot 3! + 0\cdot 2! + 0\cdot 1! = 1000_{(fb)}
\end{aligned}
$$

Program 4.5. for transforming a number written in base (10) based on the numeration (*fb*).

$$
FB(n) := \left|
\begin{array}{l}
return\ \ 0\ \ if\ \ n{=}0\\
k \leftarrow ifpk(n)\\
v_k \leftarrow 1\\
r \leftarrow fpi(n)\\
k \leftarrow ifpk(r)\\
while\ r > 1
\end{array}
\right.
$$

$$\begin{array}{l} \left| \begin{array}{l} \left| \begin{array}{l} v_k \leftarrow v_k + 1 \\ r \leftarrow fpi(r) \\ k \leftarrow ifpk(r) \end{array} \right. \\ v_1 \leftarrow v_1 + r \\ return\ \ reverse(v) \cdot Vb(10,last(v)) \end{array} \right. \end{array}$$

The program 4.5 uses the following user functions:

Program 4.6. provides the largest number $k-1$ for which $k! > x$.

$$ifpk(x) := \left| \begin{array}{l} for\ k \in 1..18 \\ \quad return\ k-1\ \ if\ x < k! \end{array} \right.$$

and *fpi* given by 2.61, Vb which returns the vector $(b^m\ \ b^{m-1}\ \ \ldots b^0)^{\mathrm{T}}$ and the utilitarian function Mathcad *reverse*.

4.1.5 Double Factorial Base

We defined over the set of natural numbers the following infinite base: for $k \in \mathbb{N}^*$, $f_k = k!!$, denoted (*dfb*), then 1, 2, 3, 8, 15, 48, 105, 384, 945, 3840, … . We proved that every positive integer a may be uniquely written in the *double factorial base* as:

$$a = \overline{a_m \ldots a_1 a_0}_{(dfb)} = \sum_{k=0}^{m} a_k \cdot f_k\ ,$$

with all $a_k = 0, 1, \ldots, k$, for $k \in \mathbb{N}^*$, in the following way:

$$\begin{array}{rcl}
1_{(10)} & = & 1 \cdot 1!! = 1_{(dfb)} \\
2_{(10)} & = & 1 \cdot 2!! + 0 \cdot 1!! = 10_{(dfb)} \\
3_{(10)} & = & 1 \cdot 3! + 0 \cdot 2!! + 0 \cdot 1!! = 100_{(dfb)} \\
4_{(10)} & = & 1 \cdot 3! + 0 \cdot 2!! + 1 \cdot 1!! = 101_{(dfb)} \\
5_{(10)} & = & 1 \cdot 3! + 1 \cdot 2!! + 0 \cdot 1!! = 110_{(dfb)} \\
6_{(10)} & = & 2 \cdot 3! + 0 \cdot 2!! + 0 \cdot 1!! = 200_{(dfb)} \\
7_{(10)} & = & 2 \cdot 3! + 0 \cdot 2!! + 1 \cdot 1!! = 201_{(dfb)} \\
8_{(10)} & = & 1 \cdot 4!! + 0 \cdot 3! + 0 \cdot 2!! + 0 \cdot 1!! = 1000_{(dfb)} \\
9_{(10)} & = & 1 \cdot 4!! + 0 \cdot 3! + 0 \cdot 2!! + 1 \cdot 1!! = 1001_{(dfb)} \\
10_{(10)} & = & 1 \cdot 4!! + 0 \cdot 3! + 1 \cdot 2!! + 0 \cdot 1!! = 1010_{(dfb)}
\end{array}$$

and so on 1100, 1101, 1110, 1200, 10000, 10001, 10010, 10100, 10101, 10110, 10200, 10201, 11000, 11001, 11010, 11100, 11101, 11110, 11200, 20000, 20001, 20010, 20100, 20101, 20110, 20200, ...

The programs transforming the numbers in base (10) based on (*dfb*) are:

Program 4.7. for the determination of inferior double factorial part.

$$idfp(x) := \left| \begin{array}{l} return\ "undefined"\ \ if\ \ x<0 \vee x>kf(28,2) \\ for\ k \in 1..28 \\ \quad return\ \ kf(k-1,2)\ \ if\ \ x<kf(k,2) \\ return\ \ "Error." \end{array} \right.$$

Note that the number $28!! = kf(28,2) = 14283291230208$ is smaller than 10^{16}.

Function 4.8. calculates the difference between x and the inferior double factorial part,

$$dfpi(x) = x - idfp(x)\ .$$

Program 4.9. for determining $k-1$ for which $x < k!!$.

$$idfpk(x) := \left| \begin{array}{l} return\ \ "undefined"\ \ if\ \ x<0 \vee x>kf(28,2) \\ for\ \ k \in 1..28 \\ \quad return\ k-1\ if\ x<kf(k,2) \\ return\ \ "Error." \end{array} \right.$$

Program 4.10. for transforming a number written in base (10) based on numeration (*dfb*).

$$DFB(n) := \left| \begin{array}{l} return\ \ 0\ \ if\ \ n{=}0 \\ k \leftarrow idfpk(n) \\ v_k \leftarrow 1 \\ r \leftarrow dfpi(n) \\ k \leftarrow idfpk(r) \\ while\ r>1 \\ \quad \left| \begin{array}{l} v_k \leftarrow v_k + 1 \\ r \leftarrow dfpi(r) \\ k \leftarrow idfpk(r) \end{array} \right. \\ v_1 \leftarrow v_1 + r \\ return\ \ reverse(v) \cdot Vb(10, last(v)) \end{array} \right.$$

The program 4.10 calls the function Mathcad *reverse* and the program *Vb* which provides the vector $(b^m\ b^{m-1}\ \ldots b^0)^{\mathrm{T}}$.

4.1.6 Triangular Base

Numbers written in the *triangular base*, defined as follows:

$$t_k = \frac{k(k+1)}{2} \ ,$$

for $k \in \mathbb{N}^*$, denoted (tb), then 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, ... We proved that every positive integer a may be uniquely written in the *triangular base* as:

$$a = \overline{a_m \dots a_1 a_0}_{(tb)} = \sum_{k=0}^{m} a_k \cdot t_k \ ,$$

with all $a_k = 0, 1, \dots, k$, for $k \in \mathbb{N}^*$.

The series of natural numbers from 1 to 36 in base (tb) is:1, 2, 10, 11, 12, 100, 101, 102, 110, 1000, 1001, 1002, 1010, 1011, 10000, 10001, 10002, 10010, 10011, 10012, 100000, 100001, 100002, 100010, 100011, 100012, 100100, 1000000, 1000001, 1000002, 1000010, 1000011, 1000012, 1000100, 1000101, 10000000 .

4.1.7 Quadratic Base

Numbers written in the *quadratic base*, defined as follows:

$$q_k = \frac{k(k+1)(2k+1)}{6} \ ,$$

for $k \in \mathbb{N}^*$, denoted ((qb), then 1, 5, 14, 30, 55, 91, 140, 204, 285, 385, ... We proved that every positive integer a may be uniquely written in the *quadratic base* as:

$$a = \overline{a_m \dots a_1 a_0}_{(qb)} = \sum_{k=0}^{m} a_k \cdot q_k \ ,$$

with all $a_k = 0, 1, \dots, k$, for $k \in \mathbb{N}^*$.

The series of natural numbers from 1 to 36 in base (qb) is: 1, 2, 3, 4, 10, 11, 12, 13, 14, 20, 21, 22, 23, 100, 101, 102, 103, 104, 110, 111, 112, 113, 114, 120, 121, 122, 123, 200, 201, 1000, 1001, 1002, 1003, 1004, 1010, 1011 .

4.1.8 Pentagon Base

Numbers written in the *pentagon base*, defined as follows:

$$pe_k = \frac{k^2(k+1)^2}{4} \ ,$$

for $k \in \mathbb{N}^*$, denoted (peb), then 1, 9, 36, 100, 225, 441, 784, 1296, 2025, 3025, $\ldots$. We proved that every positive integer a may be uniquely written in the *pentagon base* as:

$$a = \overline{a_m \ldots a_1 a_0}_{(\text{peb})} = \sum_{k=0}^{m} a_k \cdot pe_k \ ,$$

with all $a_k = 0, 1, \ldots, k$, for $k \in \mathbb{N}^*$.

The series of natural numbers from 1 to 100 in base (peb) is: 1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15, 16, 17, 18, 20, 21, 22, 23, 24, 25, 26, 27, 28, 30, 31, 32, 33, 34, 35, 36, 37, 38, 100, 101, 102, 103, 104, 105, 106, 107, 108, 110, 111, 112, 113, 114, 115, 116, 117, 118, 120, 121, 122, 123, 124, 125, 126, 127, 128, 130, 131, 132, 133, 134, 135, 136, 137, 138, 200, 201, 202, 203, 204, 205, 206, 207, 208, 210, 211, 212, 213, 214, 215, 216, 217, 218, 220, 221, 222, 223, 224, 225, 226, 227, 228, 230, 1000 .

4.1.9 Fibonacci Base

Numbers written in the *Fibonacci base*, defined as follows:

$$f_{k+2} = f_{k+1} + f_k \ ,$$

with $f_1 = 1$, $f_2 = 2$, for $k \in \mathbb{N}^*$, denoted (Fb), then 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, $\ldots$. We proved that every positive integer a may be uniquely written in the *Fibonacci base* as:

$$a = \overline{a_m \ldots a_1 a_0}_{(Fb)} = \sum_{k=0}^{m} a_k \cdot f_k \ ,$$

with all $a_k = 0, 1, \ldots, k$, for $k \in \mathbb{N}^*$. With programs like programs 4.7, 4.8, 4.9 and 4.10 we can generate natural numbers up to 50 in base (Fb): 1, 10, 100, 101, 1000, 1001, 1010, 10000, 10001, 10010, 10100, 10101, 100000, 100001, 100010, 100100, 100101, 101000, 101001, 101010, 1000000, 1000001, 1000010, 1000100, 1000101, 1001000, 1001001, 1001010, 1010000, 1010001, 1010010, 1010100, 1010101, 10000000, 10000001, 10000010, 10000100, 10000101, 10001000, 10001001, 10001010, 10010000, 10010001, 10010010, 10010100, 10010101, 10100000, 10100001, 10100010, 10100100 .

4.1.10 Tribonacci Base

Numbers written in the *Tribonacci base*, defined as follows:

$$t_{k+3} = t_{k+2} + t_{k+1} + t_k \ ,$$

with $t_1 = 1$, $t_2 = 2$, $t_3 = 3$, for $k \in \mathbb{N}^*$, denoted (Tb), then 1, 2, 3, 6, 11, 20, 37, 68, 125, 230 … . We proved that every positive integer a may be uniquely written in the *Tribonacci base* as:

$$a = \overline{a_m \dots a_1 a_0}_{(Tb)} = \sum_{k=0}^{m} a_k \cdot t_k \ ,$$

with all $a_k = 0, 1, \dots, k$, for $k \in \mathbb{N}^*$. With programs like programs 4.7, 4.8, 4.9 and 4.10 we can generate natural numbers up to 50 in base (Tb): 1, 10, 100, 101, 110, 1000, 1001, 1010, 1100, 1101, 10000, 10001, 10010, 10100, 10101, 10110, 11000, 11001, 11010, 100000, 100001, 100010, 100100, 100101, 100110, 101000, 101001, 101010, 101100, 101101, 110000, 110001, 110010, 110100, 110101, 110110, 1000000, 1000001, 1000010, 1000100, 1000101, 1000110, 1001000, 1001001, 1001010, 1001100, 1001101, 1010000, 1010001, 1010010 .

4.2 Smarandache Numbers

Smaranadche numbers are generated with commands: $n := 1..65$, $S(n,1) =$, where the function S is given by 2.69: 1, 2, 3, 4, 5, 3, 7, 4, 6, 5, 11, 4, 13, 7, 5, 6, 17, 6, 19, 5, 7, 11, 23, 4, 10, 13, 9, 7, 29, 5, 31, 8, 11, 17, 7, 6, 37, 19, 13, 5, 41, 7, 43, 11, 5, 23, 47, 6, 14, 10, 17, 13, 53, 9, 11, 7, 19, 29, 59, 5, 61, 31, 7, 8, 13, … .

4.3 Smarandache Quotients

4.3.1 Smarandache Quotients of First Kind

For each n to find the smallest k such that $n \cdot k$ is a factorial number.

Program 4.11. calculation of the number Smarandache quotient.

$$SQ(n,k) := \left| \begin{array}{l} for\ m \in 1..n \\ return\ \dfrac{kf(m,k)}{n}\ if\ mod\big(kf(m,k),n\big)=0 \end{array} \right.$$

The program *kf*, 2.15, calculates multifactorial.

The first 30 numbers *Smarandache quotients of first kind* are: 1, 1, 2, 6, 24, 1, 720, 3, 80, 12, 3628800, 2, 479001600, 360, 8, 45, 20922789888000, 40, 6402373705728000, 6, 240, 1814400, 1124000727777607680000, 1, 145152, 239500800, 13440, 180, 304888344611713860501504000000, 4 .

These numbers were obtained using the commands: $n := 1..30$, $sq1_n := SQ(n,1)$ and $sq1^{\mathrm{T}} \rightarrow$, where SQ is the program 4.11.

4.3.2 Smarandache Quotients of Second Kind

For each n to find the smallest k such that $n \cdot k$ is a double factorial number. The first 30 numbers *Smarandache quotients of second kind* are: 1, 1, 1, 2, 3, 8, 15, 1, 105, 384, 945, 4, 10395, 46080, 1, 3, 2027025, 2560, 34459425, 192, 5, 3715891200, 13749310575, 2, 81081, 1961990553600, 35, 23040, 213458046676875, 128 .

These numbers were obtained using the commands: $n := 1..30$, $sq2_n := SQ(n,2)$ and $sq2^{\mathrm{T}} \rightarrow$, where SQ is the program 4.11.

4.3.3 Smarandache Quotients of Third Kind

For each n to find the smallest k such that $n \cdot k$ is a triple factorial number. The first 30 numbers *Smarandache quotients of third kind* are: 1, 1, 1, 1, 2, 3, 4, 10, 2, 1, 80, 162, 280, 2, 1944, 5, 12320, 1, 58240, 4, 524880, 40, 4188800, 81, 167552, 140, 6, 1, 2504902400, 972 .

These numbers were obtained using the commands: $n := 1..30$, $sq3_n := SQ(n,3)$ and $sq3^{\mathrm{T}} \rightarrow$, where SQ is the program 4.11.

4.4 Primitive Numbers

4.4.1 Primitive Numbers of Power 2

$S2(n)$ is the smallest integer such that $S2(n)!$ is divisible by 2^n. The first primitive numbers (of power 2) are: 2, 4, 4, 6, 8, 8, 8, 10, 12, 12, 14, 16, 16, 16, 16, 18, 20, 20, 22, 24, 24, 24, 26, 28, 28, 30, 32, 32, 32, 32, 32, 34, 36, 36, 38, 40, 40, 40, 42, 44, 44, 46, 48, 48, 48, 48, 50, 52, 52, 54, 56, 56, 56, 58, 60, 60, 62, 64, 64, 64, 64, 64, 64, 66, $\ldots$. This sequence was generated with the program *Spk*, given by 4.12.

Curious property: This is the sequence of even numbers, each number being repeated as many times as its exponent (of power 2) is.

This is one of irreducible functions, noted $S2(k)$, which helps to calculate the Smarandache function, 2.69.

4.4.2 Primitive Numbers of Power 3

$S3(n)$ is the smallest integer such that $S3(n)!$ is divisible by 3^n. The first primitive numbers (of power 3) obtain with command $Spk(n,3) \rightarrow$ 3, 6, 9, 9, 12, 15, 18, 18, 21, 24, 27, 27, 27, 30, 33, 36, 36, 39, 42, 45, 45, 48, 51, 54, 54, 54,57, 60, 63, 63, 66, 69, 72, 72, 75, 78, 81, 81, 81, 81, 84, 87, 90, 90, 93, 96, 99, 99, 102, 105, 108, 108, 108, 111, $\ldots$. The program *Spk* is given by 4.12.

Curious property: this is the sequence of multiples of 3, each number being repeated as many times as its exponent (of power 3) is.

This is one of irreducible functions, noted *S3*(k), which helps to calculate the Smarandache function, 2.69.

4.4.3 Primitive Numbers of Power Prime

Let $p \in \mathbb{P}_{\geq 2}$, then $m = Spk(n, p, k)$ is the smallest integer such that $m!!\ldots!$ ($k-$factorial) is divisible by p^n.

Program 4.12. for generated primitive numbers of power p and factorial k.

$$Spk(n,p,k) := \left| \begin{array}{l} for\ m \in 1..n \cdot p \\ \quad return\ m\ if\ mod\big(kf(m,k), p^n\big) = 0 \\ return\ -1 \end{array} \right.$$

Proposition 4.13. *For every $m > n \cdot p$, $p^n \nmid m!!\ldots!$ ($k-$factorial).*

Proof. Case $m!$. Let $m = (n+1)\cdot$, then $m! = 1 \cdots p \cdots 2p \cdots (n+1)p$, i.e. we have $n+1$ of p in factorial, then $p^n \nmid m!$.

Case $m!!$. Let $m = (n+1)p$, where n is odd, then $m!! = 1 \cdot 3 \cdots p \cdots 3p \cdots n\cdot$, i.e. number of p in the factorial product is $< n$, then $p^n \nmid m!!$. If n is even, then $m!! = 1 \cdot 3 \cdots p \cdots 3p \cdots (n+1)p$ and now number of p in the factorial product is $< n$, then $p^n \nmid m!!$.

Cases $m!!!$, ..., $m!\ldots!$ (k times). These cases prove analogous. □

We consider command $n := 1..40$, then

$\boldsymbol{Spk}(n,2,1) \rightarrow$ 2, 4, 4, 6, 8, 8, 8, 10, 12, 12, 14, 16, 16, 16, 16, 18, 20, 20, 22, 24, 24, 24, 26, 28, 28, 30, 32, 32, 32, 32, 32, 34, 36, 36, 38, 40, 40, 40, 42, 44 .

$\boldsymbol{Spk}(n,2,2) \rightarrow$ 2, 4, 4, 6, 8, 8, 8, 10, 12, 12, 14, 16, 16, 16, 16, 18, 20, 20, 22, 24, 24, 24, 26, 28, 28, 30, 32, 32, 32, 32, 32, 34, 36, 36, 38, 40, 40, 40, 42, 44 .

$\boldsymbol{Spk}(n,2,3) \rightarrow$ 2, 4, −1, 8, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1 . In general, this case does not make sense.

$\boldsymbol{Spk}(n,2,4) \rightarrow$ 2, 4, −1, 8, 8, 12, 12, 16, 16, 16, 16, 20, 20, 24, 24, 24, 28, 28, 32, 32, 32, 32, 32, 36, 36, 40, 40, 40, 44, 44, 48, 48, 48, 48, 52, 52, 56, 56, 56, 60 .

$\boldsymbol{Spk}(n,3,1) \rightarrow$ 3, 6, 9, 9, 12, 15, 18, 18, 21, 24, 27, 27, 27, 30, 33, 36, 36, 39, 42, 45, 45, 48, 51, 54, 54, 54, 57, 60, 63, 63, 66, 69, 72, 72, 75, 78, 81, 81, 81, 81 .

Spk$(n,3,2) \to$ 3, −1, 9, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1 . In general, this case does not make sense.

Spk$(n,3,3) \to$ 3, 6, 9, 9, 12, 15, 18, 18, 21, 24, 27, 27, 27, 30, 33, 36, 36, 39, 42, 45, 45, 48, 51, 54, 54, 54, 57, 60, 63, 63, 66, 69, 72, 72, 75, 78, 81, 81, 81, 81 .

Spk$(n,5,1) \to$ 5, 10, 15, 20, 25, 25, 30, 35, 40, 45, 50, 50, 55, 60, 65, 70, 75, 75, 80, 85, 90, 95, 100, 100, 105, 110, 115, 120, 125, 125, 125, 130, 135, 140, 145, 150, 150, 155, 160, 165 .

Spk$(n,5,2) \to$ 5, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1 . In general, this case does not make sense.

Spk$(n,5,3) \to$ 5, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1 . In general, this case does not make sense.

Spk$(n,5,4) \to$ 5, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1, −1 . In general, this case does not make sense.

Spk$(n,5,5) \to$ 5, 10, 15, 20, 25, 25, 30, 35, 40, 45, 50, 50, 55, 60, 65, 70, 75, 75, 80, 85, 90, 95, 100, 100, 105, 110, 115, 120, 125, 125, 125, 130, 135, 140, 145, 150, 150, 155, 160, 165 .

4.5 $m-$Power Residues

4.5.1 Square Residues

For $n \in \mathbb{N}^*$ square residues (denoted by s_r) is: if $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_s^{\alpha_s}$, then $s_r(n) = p_1^{min\{1,\alpha_1\}} \cdot p_2^{min\{1,\alpha_2\}} \cdots p_s^{min\{1,\alpha_s\}}$. The sequence numbers square residues is: 1, 2, 3, 2, 5, 6, 7, 2, 3, 10, 11, 6, 13, 14, 15, 2, 17, 6, 19, 10, 21, 22, 23, 6, 5, 26, 3, 14, 29, 30, 31, 2, 33, 34, 35, 6, 37, 38, 39, 10, 41, 42, 43, 22, 15, 46, 47, 6, 7, 10, 51, 26, 53, 6, 14, 57, 58, 59, 30, 61, 62, 21, … .

4.5.2 Cubic Residues

For $n \in \mathbb{N}^*$ cubic residues (denoted by c_r) is: if $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_s^{\alpha_s}$, then $c_r(n) = p_1^{\min\{2,\alpha_1\}} \cdot p_2^{\min\{2,\alpha_2\}} \cdots p_s^{\min\{2,\alpha_s\}}$. The sequence numbers cubic residues is: 1, 2, 3, 4, 5, 6, 7, 4, 9, 10, 11, 12, 13, 14, 15, 4, 17, 18, 19, 20, 21, 22, 23, 12, 25,

26, 9, 28, 29, 30, 31, 4, 33, 34, 35, 36, 37, 38, 39, 20, 41, 42, 43, 44, 45, 46, 47, 12, 49, 50, 51, 52, 53, 18, 55, 28, ...

4.5.3 $m-$Power Residues

For $n \in \mathbb{N}^*$ m–power residues (denoted by m_r) is: if $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_s^{\alpha_s}$, then $m_r(n) = p_1^{\min\{m-1,\alpha_1\}} \cdot p_2^{\min\{m-1,\alpha_2\}} \cdots p_s^{\min\{m-1,\alpha_s\}}$.

4.6 Exponents of Power m

4.6.1 Exponents of Power 2

For $n \in \mathbb{N}^*$, $e_2(n)$ is the largest exponent of power 2 which divides n or $e_2(n) = k$ if 2^k divides n but 2^{k+1} does not.

Program 4.14. for calculating the number $e_b(n)$.

```
Exp(b,n) := | a ← 0
            | k ← 1
            | while b^k ≤ n
            |    | a ← k if  mod (n, b^k)=0
            |    | k ← k + 1
            | return  a
```

For $n = 1, 2, \ldots, 200$ and $Exp(2, n) =$, the program 4.14, one obtains: 0, 1, 0, 2, 0, 1, 0, 3, 0, 1, 0, 2, 0, 1, 0, 4, 0, 1, 0, 2, 0, 1, 0, 3, 0, 1, 0, 2, 0, 1, 0, 5, 0, 1, 0, 2, 0, 1, 0, 3, 0, 1, 0, 2, 0, 1, 0, 4, 0, 1, 0, 2, 0, 1, 0, 3, 0, 1, 0, 2, 0, 1, 0, 6, 0, 1, 0, 2, 0, 1, 0, 3, 0, 1, 0, 2, 0, 1, 0, 4, 0, 1, 0, 2, 0, 1, 0, 3, 0, 1, 0, 2, 0, 1, 0, 5, 0, 1, 0, 2, 0, 1, 0, 3, 0, 1, 0, 2, 0, 1, 0, 4, 0, 1, 0, 2, 0, 1, 0, 3, 0, 1, 0, 2, 0, 1, 0, 7, 0, 1, 0, 2, 0, 1, 0, 3, 0, 1, 0, 2, 0, 1, 0, 4, 0, 1, 0, 2, 0, 1, 0, 3, 0, 1, 0, 2, 0, 1, 0, 5, 0, 1, 0, 2, 0, 1, 0, 3, 0, 1, 0, 2, 0, 1, 0, 4, 0, 1, 0, 2, 0, 1, 0, 3, 0, 1, 0, 2, 0, 1, 0, 6, 0, 1, 0, 2, 0, 1, 0, 3.

4.6.2 Exponents of Power 3

For $n \in \mathbb{N}^*$, $e_3(n)$ is the largest exponent of power 3 which divides n or $e_3(n) = k$ if 3^k divides n but 3^{k+1} does not.

For $n = 1, 2, \ldots, 200$ and $Exp(3, n) =$, the program 4.14, one obtains: 0, 0, 1, 0, 0, 1, 0, 0, 2, 0, 0, 1, 0, 0, 1, 0, 0, 2, 0, 0, 1, 0, 0, 1, 0, 0, 3, 0, 0, 1, 0, 0, 1, 0, 0, 2, 0, 0, 1, 0, 0, 1, 0, 0, 2, 0, 0, 1, 0, 0, 1, 0, 0, 3, 0, 0, 1, 0, 0, 1, 0, 0, 2, 0, 0, 1, 0, 0, 1, 0, 0, 2, 0, 0, 1, 0, 0, 1, 0, 0, 4, 0, 0, 1, 0, 0, 1, 0, 0, 2, 0, 0, 1, 0, 0, 1, 0, 0, 2, 0, 0, 1, 0, 0, 1, 0, 0, 3, 0,

0, 1, 0, 0, 1, 0, 0, 2, 0, 0, 1, 0, 0, 1, 0, 0, 2, 0, 0, 1, 0, 0, 1, 0, 0, 3, 0, 0, 1, 0, 0, 1, 0, 0, 2, 0, 0, 1, 0, 0, 1, 0, 0, 2, 0, 0, 1, 0, 0, 1, 0, 0, 4, 0, 0, 1, 0, 0, 1, 0, 0, 2, 0, 0, 1, 0, 0, 1, 0, 0, 2, 0, 0, 1, 0, 0, 1, 0, 0, 3, 0, 0, 1, 0, 0, 1, 0, 0, 2, 0, 0.

4.6.3 Exponents of Power b

For $n \in \mathbb{N}^*$, $e_b(n)$ is the largest exponent of power b which divides n or $e_b(n) = k$ if b^k divides n but b^{k+1} does not.

For $n = 1, 2, \ldots, 60$ and $Exp(5, n) =$, the program 4.14, one obtains: 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 2, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 2, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1.

4.7 Almost Prime

4.7.1 Almost Primes of First Kind

Let $a_1 \geq 2$, and for $k \geq 1$, a_{k+1} is the smallest number that is not divisible by any of the previous terms (of the sequence) a_1, a_2, ..., a_k. If one starts by $a_1 = 2$, it obtains the complete prime sequence and only it.

If one starts by $a_1 > 2$, it obtains after a rank r, where $a_r = spp(a_1)^2$ with $spp(x)$, 2.42, the strictly superior prime part of x, i.e. the largest prime strictly less than x, the prime sequence:

- between a_1 and a_r, the sequence contains all prime numbers of this interval and some composite numbers;
- from a_{r+1} and up, the sequence contains all prime numbers greater than a_r and no composite numbers.

Program 4.15. for generating the numbers *almost primes of first kind* de la n la L.

```
AP1(n,L) := | j ← 1
            | a_j ← n
            | for m ∈ n+1..L
            |   | sw ← 0
            |   | for k ∈ 1..j
            |   |   if  mod(m, a_k)=0
            |   |     | sw ← 1
            |   |     | break
            |   | if sw=0
            |   |   | j ← j+1
```

```
| | | a_j ← m
| return a
```

The first numbers *almost prime of first kind* given by *AP1*$(10, 10^3)$ are: 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 21, 23, 25, 27, 29, 31, 35, 37, 41, 43, 47, 49, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113, 127, 131, 137, 139, 149, 151, 157, 163, 167, 173, 179, 181, 191, 193, 197, 199, 211, 223, 227, 229, 233, 239, 241, 251, 257, 263, 269, 271, 277, 281, 283, 293, 307, 311, 313, 317, 331, 337, 347, 349, 353, 359, 367, 373, 379, 383, 389, 397, 401, 409, 419, 421, 431, 433, 439, 443, 449, 457, 461, 463, 467, 479, 487, 491, 499, 503, 509, 521, 523, 541, 547, 557, 563, 569, 571, 577, 587, 593, 599, 601, 607, 613, 617, 619, 631, 641, 643, 647, 653, 659, 661, 673, 677, 683, 691, 701, 709, 719, 727, 733, 739, 743, 751, 757, 761, 769, 773, 787, 797, 809, 811, 821, 823, 827, 829, 839, 853, 857, 859, 863, 877, 881, 883, 887, 907, 911, 919, 929, 937, 941, 947, 953, 967, 971, 977, 983, 991, 997 (See Figure 10.1).

4.7.2 Almost Prime of Second Kind

Let $a_1 \geq 2$, and for $k \geq 1$, a_{k+1} is the smallest number that is coprime (a is coprime $b \Leftrightarrow gcd(a,b) = 1$ with all of the previous terms (of the sequence), a_1, $a_2, \ldots, a_k$.

This second kind sequence merges faster to the prime numbers than the first kind sequence. If one starts by $a_1 = 2$, it obtains the complete prime sequence and only it.

If one starts by $a_1 > 2$, it obtains after a rank r, where $a_1 = p_i \cdot p_j$ with p_i and p_j prime numbers strictly less than and not dividing a_1, the prime sequence:

- between a_1 and a_r, the sequence contains all prime numbers of this interval and some composite numbers;
- from a_{r+1} and up, the sequence contains all prime numbers greater than a_r and no composite numbers.

Program 4.16. for generating the numbers *almost primes of second kind.*

```
AP2(n,L) := | j ← 1
            | a_j ← n
            | for m ∈ n+1..L
            |   | sw ← 0
            |   | for k ∈ 1..j
            |   |   if gcd(m, a_k) ≠ 1
            |   |     | sw ← 1
            |   |     | break
            |   | if sw=0
```

$$\begin{array}{l} \quad\quad j \leftarrow j+1 \\ \quad\quad a_j \leftarrow m \\ return\ \ a \end{array}$$

The first numbers almost prime of second kind given by the program *AP2*$(10, 10^3)$ are: 10, 11, 13, 17, 19, 21, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113, 127, 131, 137, 139, 149, 151, 157, 163, 167, 173, 179, 181, 191, 193, 197, 199, 211, 223, 227, 229, 233, 239, 241, 251, 257, 263, 269, 271, 277, 281, 283, 293, 307, 311, 313, 317, 331, 337, 347, 349, 353, 359, 367, 373, 379, 383, 389, 397, 401, 409, 419, 421, 431, 433, 439, 443, 449, 457, 461, 463, 467, 479, 487, 491, 499, 503, 509, 521, 523, 541, 547, 557, 563, 569, 571, 577, 587, 593, 599, 601, 607, 613, 617, 619, 631, 641, 643, 647, 653, 659, 661, 673, 677, 683, 691, 701, 709, 719, 727, 733, 739, 743, 751, 757, 761, 769, 773, 787, 797, 809, 811, 821, 823, 827, 829, 839, 853, 857, 859, 863, 877, 881, 883, 887, 907, 911, 919, 929, 937, 941, 947, 953, 967, 971, 977, 983, 991, 997 (See Figure 10.1).

4.8 Pseudo–Primes

4.8.1 Pseudo–Primes of First Kind

Definition 4.17. A number is a *pseudo–prime of first kind* if exist a permutation of the digits that is a prime number, including the identity permutation.

The matrices $Per2$, $Per3$ and $Per4$ contain all the permutations from 2, 3 and 4.

$$Per2 := \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \tag{4.1}$$

$$Per3 := \begin{pmatrix} 1 & 1 & 2 & 2 & 3 & 3 \\ 2 & 3 & 1 & 3 & 1 & 2 \\ 3 & 2 & 3 & 1 & 2 & 1 \end{pmatrix} \tag{4.2}$$

$$Per4 := \left(\begin{array}{cccccccccccccccccccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 3 & 3 & 4 & 4 & 4 & 4 & 4 & 4 \\ 2 & 2 & 3 & 3 & 4 & 4 & 1 & 1 & 3 & 3 & 4 & 4 & 1 & 1 & 2 & 2 & 4 & 4 & 1 & 1 & 2 & 2 & 3 & 3 \\ 3 & 4 & 2 & 4 & 2 & 3 & 3 & 4 & 1 & 4 & 1 & 3 & 2 & 4 & 1 & 4 & 1 & 2 & 2 & 3 & 1 & 3 & 1 & 2 \\ 4 & 3 & 4 & 2 & 3 & 2 & 4 & 3 & 4 & 1 & 3 & 1 & 4 & 2 & 4 & 1 & 2 & 1 & 3 & 2 & 3 & 1 & 2 & 1 \end{array}\right) \tag{4.3}$$

Program 4.18. for counting the primes obtained by the permutation of number's digits.

$$
PP(n,q):=\left|\begin{array}{l}
m \leftarrow nrd(n,10)\\
d \leftarrow dn(n,10)\\
np \leftarrow 1 \;\; if \;\; m{=}1\\
np \leftarrow cols(Per2) \; if \; m{=}2\\
np \leftarrow cols(Per3) \; if \; m{=}3\\
np \leftarrow cols(Per4) \; if \; m{=}4\\
sw \leftarrow 0\\
for \; j \in q..max(q,np)\\
\quad \left|\begin{array}{l}
for \; k \in 1..m\\
\quad \left|\begin{array}{l}
pd \leftarrow d \;\; if \; m{=}1\\
pd_k \leftarrow d_{(Per2_{k,j})} \;\; if \; m{=}2\\
pd_k \leftarrow d_{(Per3_{k,j})} \;\; if \; m{=}3\\
pd_k \leftarrow d_{(Per4_{k,j})} \;\; if \; m{=}4
\end{array}\right.\\
nn \leftarrow pd \cdot Vb(10,m)\\
sw \leftarrow sw+1 \; if \; TS(nn){=}1
\end{array}\right.\\
return \;\; sw
\end{array}\right.
$$

The program uses the subprograms: *nrd* given by 2.1, *dn* given by 2.2, $Vb(b,m)$ which returns the vector $(b^m \; b^{m-1} \; \ldots b^0)^{\mathrm{T}}$, *TS* Smarandache primality test defined at 1.5. Also, the program entails the matrices (4.1), (4.2) and (4.3) which contain all the permutation of sets $\{1,2\}$, $\{1,2,3\}$ and $\{1,2,3,4\}$.

The first 457 of numbers *pseudo-prime of first kinds* are: 2, 3, 5, 7, 11, 13, 14, 16, 17, 19, 20, 23, 29, 30, 31, 32, 34, 35, 37, 38, 41, 43, 47, 50, 53, 59, 61, 67, 70, 71, 73, 74, 76, 79, 83, 89, 91, 92, 95, 97, 98, 101, 103, 104, 106, 107, 109, 110, 112, 113, 115, 118, 119, 121, 124, 125, 127, 128, 130, 131, 133, 134, 136, 137, 139, 140, 142, 143, 145, 146, 149, 151, 152, 154, 157, 160, 163, 164, 166, 167, 169, 170, 172, 173, 175, 176, 179, 181, 182, 188, 190, 191, 193, 194, 196, 197, 199, 200, 203, 209, 211, 214, 215, 217, 218, 223, 227, 229, 230, 232, 233, 235, 236, 238, 239, 241, 251, 253, 257, 263, 269, 271, 272, 275, 277, 278, 281, 283, 287, 289, 290, 292, 293, 296, 298, 299, 300, 301, 302, 304, 305, 307, 308, 310, 311, 313, 314, 316, 317, 319, 320, 322, 323, 325, 326, 328, 329, 331, 332, 334, 335, 337, 338, 340, 341, 343, 344, 346, 347, 349, 350, 352, 353, 356, 358, 359, 361, 362, 364, 365, 367, 368, 370, 371, 373, 374, 376, 377, 379, 380, 382, 383, 385, 386, 388, 389, 391, 392, 394, 395, 397, 398, 401, 403, 407, 409, 410, 412, 413, 415, 416, 419, 421, 430, 431, 433, 434, 436, 437, 439, 443, 449, 451, 457, 461, 463, 467, 470, 473, 475, 476, 478, 479, 487, 490, 491, 493, 494, 497, 499, 500, 503, 509, 511, 512, 514, 517, 521, 523, 527, 530, 532, 533, 536, 538, 539, 541, 547, 557, 563, 569, 571, 572, 574, 575, 577, 578, 583, 587, 589, 590,

593, 596, 598, 599, 601, 607, 610, 613, 614, 616, 617, 619, 623, 629, 631, 632, 634, 635, 637, 638, 641, 643, 647, 653, 659, 661, 670, 671, 673, 674, 677, 679, 683, 691, 692, 695, 697, 700, 701, 703, 704, 706, 709, 710, 712, 713, 715, 716, 719, 721, 722, 725, 727, 728, 730, 731, 733, 734, 736, 737, 739, 740, 743, 745, 746, 748, 749, 751, 752, 754, 755, 757, 758, 760, 761, 763, 764, 767, 769, 772, 773, 775, 776, 778, 779, 782, 784, 785, 787, 788, 790, 791, 793, 794, 796, 797, 799, 803, 809, 811, 812, 818, 821, 823, 827, 829, 830, 832, 833, 835, 836, 838, 839, 847, 853, 857, 859, 863, 872, 874, 875, 877, 878, 881, 883, 887, 890, 892, 893, 895, 901, 902, 904, 905, 907, 908, 910, 911, 913, 914, 916, 917, 919, 920, 922, 923, 926, 928, 929, 931, 932, 934, 935, 937, 938, 940, 941, 943, 944, 947, 949, 950, 953, 956, 958, 959, 961, 962, 965, 967, 970, 971, 973, 974, 976, 977, 979, 980, 982, 983, 985, 991, 992, 994, 995, 997 . This numbers obtain with command *APP1*$(2,999)$, where program *APP1* is:

Program 4.19. for displaying the Pseudo–Primes of First Kind numbers.

$$
APP1(a,b) := \left|
\begin{array}{l}
j \leftarrow 1 \\
for\ n \in a..b \\
\quad if\ PP(n,1) \geq 1 \\
\quad \left| \begin{array}{l} pp_j \leftarrow n \\ j \leftarrow j+1 \end{array} \right. \\
return\ \ pp
\end{array}
\right.
$$

4.8.2 Pseudo–Primes of Second Kind

Definition 4.20. A composite number is a *pseudo–prime of second kind* if exist a permutation of the digits that is a prime number.

The first 289 of numbers *pseudo–prime of second kinds* are: 14, 16, 20, 30, 32, 34, 35, 38, 50, 70, 74, 76, 91, 92, 95, 98, 104, 106, 110, 112, 115, 118, 119, 121, 124, 125, 128, 130, 133, 134, 136, 140, 142, 143, 145, 146, 152, 154, 160, 164, 166, 169, 170, 172, 175, 176, 182, 188, 190, 194, 196, 200, 203, 209, 214, 215, 217, 218, 230, 232, 235, 236, 238, 253, 272, 275, 278, 287, 289, 290, 292, 296, 298, 299, 300, 301, 302, 304, 305, 308, 310, 314, 316, 319, 320, 322, 323, 325, 326, 328, 329, 332, 334, 335, 338, 340, 341, 343, 344, 346, 350, 352, 356, 358, 361, 362, 364, 365, 368, 370, 371, 374, 376, 377, 380, 382, 385, 386, 388, 391, 392, 394, 395, 398, 403, 407, 410, 412, 413, 415, 416, 430, 434, 436, 437, 451, 470, 473, 475, 476, 478, 490, 493, 494, 497, 500, 511, 512, 514, 517, 527, 530, 532, 533, 536, 538, 539, 572, 574, 575, 578, 583, 589, 590, 596, 598, 610, 614, 616, 623, 629, 632, 634, 635, 637, 638, 670, 671, 674, 679, 692, 695, 697, 700, 703, 704, 706, 710, 712, 713, 715, 716, 721, 722, 725, 728, 730, 731, 734, 736, 737, 740, 745, 746, 748, 749, 752, 754, 755, 758, 760, 763, 764, 767, 772, 775, 776, 778, 779, 782, 784, 785, 788, 790, 791, 793, 794, 796, 799, 803, 812, 818, 830, 832, 833, 835, 836, 838, 847, 872, 874, 875, 878, 890,

892, 893, 895, 901, 902, 904, 905, 908, 910, 913, 914, 916, 917, 920, 922, 923, 926, 928, 931, 932, 934, 935, 938, 940, 943, 944, 949, 950, 956, 958, 959, 961, 962, 965, 970, 973, 974, 976, 979, 980, 982, 985, 992, 994, 995 . This numbers obtain with command *APP2*$(2,999)$, where program *APP2* is:

Program 4.21. of displaying the Pseudo–Primes of Second Kind numbers.

$$
APP2(a,b) := \left|
\begin{array}{l}
j \leftarrow 1 \\
for\ n \in a..b \\
\quad if\ TS(n){=}0 \land PP(n,1) \geq 1 \\
\quad\quad \left| \begin{array}{l} pp_j \leftarrow n \\ j \leftarrow j+1 \end{array} \right. \\
return\ \ pp
\end{array}
\right.
$$

4.8.3 Pseudo–Primes of Third Kind

Definition 4.22. A number is a *pseudo–prime of third kind* if exist a nontrivial permutation of the digits that is a prime number.

The first 429 of numbers *pseudo-prime of third kinds* are: 11, 13, 14, 16, 17, 20, 30, 31, 32, 34, 35, 37, 38, 50, 70, 71, 73, 74, 76, 79, 91, 92, 95, 97, 98, 101, 103, 104, 106, 107, 109, 110, 112, 113, 115, 118, 119, 121, 124, 125, 127, 128, 130, 131, 133, 134, 136, 137, 139, 140, 142, 143, 145, 146, 149, 151, 152, 154, 157, 160, 163, 164, 166, 167, 169, 170, 172, 173, 175, 176, 179, 181, 182, 188, 190, 191, 193, 194, 196, 197, 199, 200, 203, 209, 211, 214, 215, 217, 218, 223, 227, 229, 230, 232, 233, 235, 236, 238, 239, 241, 251, 253, 271, 272, 275, 277, 278, 281, 283, 287, 289, 290, 292, 293, 296, 298, 299, 300, 301, 302, 304, 305, 307, 308, 310, 311, 313, 314, 316, 317, 319, 320, 322, 323, 325, 326, 328, 329, 331, 332, 334, 335, 337, 338, 340, 341, 343, 344, 346, 347, 349, 350, 352, 353, 356, 358, 359, 361, 362, 364, 365, 367, 368, 370, 371, 373, 374, 376, 377, 379, 380, 382, 383, 385, 386, 388, 389, 391, 392, 394, 395, 397, 398, 401, 403, 407, 410, 412, 413, 415, 416, 419, 421, 430, 433, 434, 436, 437, 439, 443, 449, 451, 457, 461, 463, 467, 470, 473, 475, 476, 478, 479, 490, 491, 493, 494, 497, 499, 500, 503, 509, 511, 512, 514, 517, 521, 527, 530, 532, 533, 536, 538, 539, 547, 557, 563, 569, 571, 572, 574, 575, 577, 578, 583, 587, 589, 590, 593, 596, 598, 599, 601, 607, 610, 613, 614, 616, 617, 619, 623, 629, 631, 632, 634, 635, 637, 638, 641, 643, 647, 653, 659, 661, 670, 671, 673, 674, 677, 679, 683, 691, 692, 695, 697, 700, 701, 703, 704, 706, 709, 710, 712, 713, 715, 716, 719, 721, 722, 725, 727, 728, 730, 731, 733, 734, 736, 737, 739, 740, 743, 745, 746, 748, 749, 751, 752, 754, 755, 757, 758, 760, 761, 763, 764, 767, 769, 772, 773, 775, 776, 778, 779, 782, 784, 785, 787, 788, 790, 791, 793, 794, 796, 797, 799, 803, 809, 811, 812, 818, 821, 823, 830, 832, 833, 835, 836, 838, 839, 847, 857, 863, 872, 874, 875, 877, 878, 881, 883, 887, 890, 892, 893, 895, 901, 902, 904, 905, 907, 908, 910, 911, 913, 914, 916,

917, 919, 920, 922, 923, 926, 928, 929, 931, 932, 934, 935, 937, 938, 940, 941, 943, 944, 947, 949, 950, 953, 956, 958, 959, 961, 962, 965, 967, 970, 971, 973, 974, 976, 977, 979, 980, 982, 983, 985, 991, 992, 994, 995, 997 . This numbers obtain with command *APP3*$(2,999)$, where program *APP3* is:

Program 4.23. of displaying the Pseudo–Primes of Third Kind numbers.

```
APP3(a,b) := | j ← 1
             | for n ∈ a..b
             |   if PP(n,2) ≥ 1 ∧ n > 10
             |     | pp_j ← n
             |     | j ← j + 1
             | return pp
```

Questions:

1. How many *pseudo–primes of third kind* are prime numbers? (We conjecture: an infinity).

2. There are primes which are not *pseudo–primes of third kind*, and the reverse: there are *pseudo–primes of third kind* which are not primes.

4.9 Permutation–Primes

4.9.1 Permutation–Primes of type 1

Let the permutations of 3

$$per3 = \begin{pmatrix} 1 & 1 & 2 & 2 & 3 & 3 \\ 2 & 3 & 1 & 3 & 1 & 2 \\ 3 & 2 & 3 & 1 & 2 & 1 \end{pmatrix} .$$

We denote $per3_k(\overline{d_1 d_2 d_3})$, $k = 1,2,\ldots,6$, a permutation of the number with the digits d_1, d_2, d_3. E.g. $per3_2(\overline{d_1 d_2 d_3}) = \overline{d_1 d_3 d_2}$. It is obvious that for a number of m digits one can apply a permutation of the order m.

Definition 4.24. We say that $n \in \mathbb{N}^*$ is a *permutation–prime of type 1* if there exists at least a permutation for which the resulted number is prime.

Program 4.25. of displaying the *permutation–primes*.

```
APP(a,b,k) := | j ← 1
              | for n ∈ a..b
              |   | sw ← PP(n,1)
```

```
| | if sw=k
| | | pp_j ← n
| | | j ← j+1
| return pp
```

The program using the subprogram *PP* given by 4.18.

There are 122 *permutation–primes of type 1* from 2 to 999: 2, 3, 5, 7, 14, 16, 19, 20, 23, 29, 30, 32, 34, 35, 38, 41, 43, 47, 50, 53, 59, 61, 67, 70, 74, 76, 83, 89, 91, 92, 95, 98, 134, 143, 145, 154, 203, 209, 230, 235, 236, 253, 257, 263, 269, 275, 278, 287, 289, 290, 296, 298, 302, 304, 308, 314, 320, 325, 326, 340, 341, 352, 358, 362, 380, 385, 403, 407, 409, 413, 415, 430, 431, 451, 470, 478, 487, 490, 514, 523, 527, 532, 538, 541, 572, 583, 589, 598, 623, 629, 632, 692, 704, 725, 728, 740, 748, 752, 782, 784, 803, 827, 829, 830, 835, 847, 853, 859, 872, 874, 892, 895, 902, 904, 920, 926, 928, 940, 958, 962, 982, 985 . This numbers are obtained with the command $APP(2,999,1)^{\mathrm{T}} =$, where *APP* is the program 4.25.

4.9.2 Permutation–Primes of type 2

Definition 4.26. We say that $n \in \mathbb{N}^*$ is a *permutation–primes of type 2* if there exists only two permutations for which the resulted numbers are primes.

There are 233 *permutation–primes of type 2* from 2 to 999: 11, 13, 17, 31, 37, 71, 73, 79, 97, 104, 106, 109, 112, 115, 121, 124, 125, 127, 128, 139, 140, 142, 146, 151, 152, 160, 164, 166, 169, 172, 182, 188, 190, 193, 196, 200, 211, 214, 215, 217, 218, 223, 227, 229, 232, 233, 238, 239, 241, 251, 271, 272, 281, 283, 292, 293, 299, 300, 305, 319, 322, 323, 328, 329, 332, 334, 335, 338, 343, 344, 346, 347, 349, 350, 353, 356, 364, 365, 367, 368, 374, 376, 377, 382, 383, 386, 388, 391, 392, 394, 401, 410, 412, 416, 421, 433, 434, 436, 437, 439, 443, 449, 457, 461, 463, 467, 473, 475, 476, 479, 493, 494, 497, 499, 500, 503, 509, 511, 512, 521, 530, 533, 536, 547, 557, 563, 569, 574, 575, 578, 587, 590, 596, 599, 601, 607, 610, 614, 616, 619, 634, 635, 637, 638, 641, 643, 647, 653, 659, 661, 670, 673, 674, 677, 679, 683, 691, 695, 697, 700, 706, 712, 721, 722, 734, 736, 737, 743, 745, 746, 749, 754, 755, 758, 760, 763, 764, 767, 769, 773, 776, 785, 788, 794, 796, 799, 809, 812, 818, 821, 823, 832, 833, 836, 838, 857, 863, 875, 878, 881, 883, 887, 890, 901, 905, 908, 910, 913, 916, 922, 923, 929, 931, 932, 934, 943, 944, 947, 949, 950, 956, 959, 961, 965, 967, 974, 976, 979, 980, 992, 994, 995, 997 . This numbers are obtained with the command $APP(2,999,2)^{\mathrm{T}} =$, where *APP* is the program 4.25.

4.9.3 Permutation–Primes of type 3

Definition 4.27. We say $n \in \mathbb{N}^*$ is a *permutation–primes of type 3* if there exists only three permutations for which the resulted numbers are primes.

There are 44 *permutation–primes of type 3* from 2 to 999: 103, 130, 136, 137, 157, 163, 167, 173, 175, 176, 301, 307, 310, 316, 317, 359, 361, 370, 371, 389, 395, 398, 517, 539, 571, 593, 613, 617, 631, 671, 703, 713, 715, 716, 730, 731, 751, 761, 839, 893, 935, 938, 953, 983. This numbers are obtained with the command $APP(2,999,3)^{\mathrm{T}} =$, where *APP* is the program 4.25.

4.9.4 Permutation–Primes of type *m*

Definition 4.28. We say $n \in \mathbb{N}^*$ is a *permutation–primes of type m* if there exists only m permutations for which the resulted numbers are primes.

There are 49 *permutation–primes of type 4* from 2 to 999: 101, 107, 110, 118, 119, 133, 149, 170, 179, 181, 191, 194, 197, 277, 313, 331, 379, 397, 419, 491, 577, 701, 709, 710, 719, 727, 739, 757, 772, 775, 778, 779, 787, 790, 791, 793, 797, 811, 877, 907, 911, 914, 917, 937, 941, 970, 971, 973, 977 . This numbers are obtained with the command $APP(2,999,4)^{\mathrm{T}} =$, where *APP* is the program 4.25.

There are not *permutation–primes of type 5* from 2 to 999, but there are 9 *permutation–primes of type 6* from 2 to 999: 113, 131, 199, 311, 337, 373, 733, 919, 991. This numbers are obtained with the command $APP(2,999,6)^{\mathrm{T}} =$, where *APP* is the program 4.25.

4.10 Pseudo–Squares

4.10.1 Pseudo–Squares of First Kind

Definition 4.29. A number is a *pseudo–square of first kind* if some permutation of the digits is a perfect square, including the identity permutation.

Of course, all perfect squares are pseudo–squares of first kind, but not the reverse!

Program 4.30. for counting the squares obtained by digits permutation.

$$PSq(n,i) := \left| \begin{array}{l} m \leftarrow nrd(n,10) \\ d \leftarrow dn(n,10) \\ np \leftarrow 1 \ \ if\ m{=}1 \\ np \leftarrow cols(Per2) \ \ if \ \ m{=}2 \\ np \leftarrow cols(Per3) \ \ if \ \ m{=}3 \end{array} \right.$$

$$\left| \begin{array}{l} np \leftarrow cols(Per4) \ \ if \ \ m{=}4 \\ sw \leftarrow 0 \\ for\ j \in i..np \\ \quad \left| \begin{array}{l} for\ k \in 1..m \\ \quad \left| \begin{array}{l} pd \leftarrow d \ \ if\ m{=}1 \\ pd_k \leftarrow d_{(Per2_{k,j})} \ \ if \ \ m{=}2 \\ pd_k \leftarrow d_{(Per3_{k,j})} \ \ if \ \ m{=}3 \\ pd_k \leftarrow d_{(Per4_{k,j})} \ \ if \ \ m{=}4 \end{array} \right. \\ nn \leftarrow pd \cdot Vb(10,m) \\ sw \leftarrow sw+1 \ \ if \ \ isp(nn){=}nn \end{array} \right. \\ return \ \ sw \end{array} \right.$$

The program uses the subprograms: *nrd* given by 2.1, *dn* given by 2.2, $Vb(b,m)$ which returns the vector $(b^m \ b^{m-1} \ \ldots b^0)^{\mathrm{T}}$ and *isp* defined at 2.48. Also, the program calls the matrices *Per2*, *Per3* and *Per4* which contains all permutations of sets $\{1,2\}$, $\{1,2,3\}$ and $\{1,2,3,4\}$.

Program 4.31. of displaying the *pseudo–squares of first kind*.

$$APSq1(a,b) := \left| \begin{array}{l} j \leftarrow 1 \\ for \ \ n \in a..b \\ \quad if \ \ PSq(n,1) \geq 1 \\ \quad \quad \left| \begin{array}{l} psq_j \leftarrow n \\ j \leftarrow j+1 \end{array} \right. \\ return \ \ psq \end{array} \right.$$

One listed all (are 121) *pseudo–squares of first kind* up to 1000: 1, 4, 9, 10, 16, 18, 25, 36, 40, 46, 49, 52, 61, 63, 64, 81, 90, 94, 100, 106, 108, 112, 121, 136, 144, 148, 160, 163,169, 180, 184, 196, 205, 211, 225, 234, 243, 250, 252, 256, 259, 265, 279, 289, 295, 297, 298, 306, 316, 324 ,342, 360, 361, 400, 406, 409, 414, 418, 423, 432, 441, 448, 460, 478, 481, 484, 487, 490, 502, 520, 522, 526, 529, 562, 567, 576, 592, 601, 603, 604, 610, 613, 619, 625, 630, 631, 640, 652, 657, 667, 675, 676, 691, 729, 748, 756, 765, 766, 784, 792, 801, 810, 814, 829, 841, 844, 847, 874, 892, 900, 904, 916, 925, 927, 928, 940, 952, 961, 972, 982, 1000. This numbers are obtained with the command $APSq1(1,10^3)^{\mathrm{T}} =$, where *APSq1* is the program 4.31.

4.10.2 Pseudo–Squares of Second Kind

Definition 4.32. A non–square number is a *pseudo–squares of second kind* if exist a permutation of the digits is a square.

Program 4.33. of displaying *pseudo–squares of second kind*.

$$APSq2(a,b) := \left| \begin{array}{l} j \leftarrow 1 \\ for\ n \in a..b \\ \quad if\ PSq(n,1) \geq 1 \wedge isp(n) \neq n \\ \quad \left| \begin{array}{l} psq_j \leftarrow n \\ j \leftarrow j+1 \end{array} \right. \\ return\ \ psq \end{array} \right.$$

Let us list all (there are 90) *pseudo–squares of second kind* up to 1000: 10, 18, 40, 46, 52, 61, 63, 90, 94, 106, 108, 112, 136, 148, 160, 163, 180, 184, 205, 211, 234, 243, 250, 252, 259, 265, 279, 295, 297, 298, 306, 316, 342, 360, 406, 409, 414, 418, 423, 432, 448, 460, 478, 481, 487, 490, 502, 520, 522, 526, 562, 567, 592, 601, 603, 604, 610, 613, 619, 630, 631, 640, 652, 657, 667, 675, 691, 748, 756, 765, 766, 792, 801, 810, 814, 829, 844, 847, 874, 892, 904, 916, 925, 927, 928, 940, 952, 972, 982, 1000 . This numbers are obtained with the command $APSq2(1,10^3)^{\mathrm{T}} =$, where *APSq2* is the program 4.33.

4.10.3 Pseudo–Squares of Third Kind

Definition 4.34. A number is a *pseudo–square of third kind* if exist a nontrivial permutation of the digits is a square.

Program 4.35. of displaying *pseudo–squares of third kind.*

$$APSq3(a,b) := \left| \begin{array}{l} j \leftarrow 1 \\ for\ n \in a..b \\ \quad if\ PSq(n,2) \geq 1 \wedge n > 9 \\ \quad \left| \begin{array}{l} psq_j \leftarrow n \\ j \leftarrow j+1 \end{array} \right. \\ return\ \ psq \end{array} \right.$$

Let us list all (there are 104) *pseudo–squares of third kind* up to 1000: 10, 18, 40, 46, 52, 61, 63, 90, 94, 100, 106, 108, 112, 121, 136, 144, 148, 160, 163, 169, 180, 184, 196, 205, 211, 225, 234, 243, 250, 252, 256, 259, 265, 279, 295, 297, 298, 306, 316, 342, 360, 400, 406, 409, 414, 418, 423, 432, 441, 448, 460, 478, 481, 484, 487, 490, 502, 520, 522, 526, 562, 567, 592, 601, 603, 604, 610, 613, 619, 625, 630, 631, 640, 652, 657, 667, 675, 676, 691, 748, 756, 765, 766, 792, 801, 810, 814, 829, 844, 847, 874, 892, 900, 904, 916, 925, 927, 928, 940, 952, 961, 972, 982, 1000 . This numbers are obtained with the command $APSq3(1,10^3)^{\mathrm{T}} =$, where *APSq3* is the program 4.35.

Question:

1. How many pseudo–squares of third kind are square numbers? We conjecture: an infinity.

2. There are squares which are not pseudo–squares of third kind, and the reverse: there are pseudo–squares of third kind which are not squares.

4.11 Pseudo–Cubes

4.11.1 Pseudo–Cubes of First Kind

Definition 4.36. A number is a *pseudo–cube of first kind* if some permutation of the digits is a cube, including the identity permutation.

Of course, all perfect cubes are *pseudo–cubes of first kind*, but not the reverse!

With programs similar to *PSq*, 4.30, *APSq1*, 4.31, *APSq2*, 4.31 and *APSq3*, 4.35 can list the pseudo–cube numbers.

Let us list all (there are 40) *pseudo–cubes of first kind* up to 1000: 1, 8, 10, 27, 46, 64, 72, 80, 100, 125, 126, 152, 162, 207, 215, 216, 251, 261, 270, 279, 297, 334, 343, 406, 433, 460, 512, 521, 604, 612, 621, 640, 702, 720, 729, 792, 800, 927, 972, 1000 .

4.11.2 Pseudo–Cubes of Second Kind

Definition 4.37. A non–cube number is a *pseudo–cube of second kind* if some permutation of the digits is a cube.

Let us list all (there are 30) pseudo–cubes of second kind up to 1000: 10, 46, 72, 80, 100, 126, 152, 162, 207, 215, 251, 261, 270, 279, 297, 334, 406, 433, 460, 521, 604, 612, 621, 640, 702, 720, 792, 800, 927, 972 .

4.11.3 Pseudo–Cubes of Third Kind

Definition 4.38. A number is a *pseudo–cube of third kind* if exist a nontrivial permutation of the digits is a cube.

Let us list all (there are 34) *pseudo–cubes of third kind* up to 1000: 10, 46, 72, 80, 100, 125, 126, 152, 162, 207, 215, 251, 261, 270, 279, 297, 334, 343, 406, 433, 460, 512, 521, 604, 612, 621, 640, 702, 720, 792, 800, 927, 972, 1000 .

Question:

1. How many pseudo–cubes of third kind are cubes? We conjecture: an infinity.

2. There are cubes which are not pseudo–cubes of third kind, and the reverse: there are pseudo–cubes of third kind which are not cubes.

4.12 Pseudo–m–Powers

4.12.1 Pseudo–m–Powers of First Kind

Definition 4.39. A number is a *pseudo–m–power of first kind* if exist a permutation of the digits is an m–power, including the identity permutation; $m \geq 2$.

4.12.2 Pseudo–m–Powers of Second kind

Definition 4.40. A non m–power number is a *pseudo–m–power of second kind* if exist a permutation of the digits is an m–power; $m \geq 2$.

4.12.3 Pseudo–m–Powers of Third Kind

Definition 4.41. A number is a *pseudo–m–power of third kind* if exist a nontrivial permutation of the digits is an m–power; $m \geq 2$.

Question:

1. How many pseudo–m–powers of third kind are m–power numbers? We conjecture: an infinity.

2. There are m–powers which are not *pseudo–m–powers of third kind*, and the reverse: there are *pseudo–m–powers of third kind* which are not m–powers.

4.13 Pseudo–Factorials

4.13.1 Pseudo–Factorials of First Kind

Definition 4.42. A number is a *pseudo-factorial of first kind* if exist a permutation of the digits is a factorial number, including the identity permutation.

One listed all *pseudo–factorials of first kind* up to 1000: 1, 2, 6, 10, 20, 24, 42, 60, 100, 102, 120, 200, 201, 204, 207, 210, 240, 270, 402, 420, 600, 702, 720, 1000, 1002, 1020, 1200, 2000, 2001, 2004, 2007, 2010, 2040, 2070, 2100, 2400, 2700, 4002, 4005, 4020, 4050, 4200, 4500, 5004, 5040, 5400, 6000, 7002, 7020, 7200 . In this list there are 37 numbers.

4.13.2 Pseudo–Factorials of Second Kind

Definition 4.43. A non–factorial number is a *pseudo–factorial of second kind* if exist a permutation of the digits is a factorial number.

One listed all *pseudo–factorials of second kind* up to 1000: 10, 20, 42, 60, 100, 102, 200, 201, 204, 207, 210, 240, 270, 402, 420, 600, 702, 1000, 1002, 1020, 1200, 2000,2001,2004, 2007, 2010, 2040, 2070, 2100, 2400, 2700, 4002, 4005, 4020, 4050, 4200, 4500, 5004, 5400, 6000, 7002, 7020, 7200 . In this list there are 31 numbers.

4.13.3 Pseudo–Factorials of Third Kind

Definition 4.44. A number is a *pseudo–factorial of third kind* if exist nontrivial permutation of the digits is a factorial number.

One listed all *pseudo–factorials of third kind* up to 1000: 10, 20, 42, 60, 100, 102, 200, 201, 204, 207, 210, 240, 270, 402, 420, 600, 702, 1000, 1002, 1020, 1200, 2000,2001,2004, 2007, 2010, 2040, 2070, 2100, 2400, 2700, 4002, 4005, 4020, 4050, 4200, 4500, 5004, 5400, 6000, 7002, 7020, 7200 . In this list there are 31 numbers.

Unfortunately, the second and third kinds of pseudo–factorials coincide.

Question:

1. How many *pseudo–factorials of third kind* are factorial numbers?

2. We conjectured: none! ... that means the*pseudo–factorials of second kind* set and *pseudo–factorials of third kind* set coincide!

4.14 Pseudo–Divisors

4.14.1 Pseudo–Divisors of First Kind

Definition 4.45. A number is a *pseudo–divisor of first kind* of n if exist a permutation of the digits is a divisor of n, including the identity permutation.

Table 4.2: Pseudo–divisor of first kind of $n \leq 12$

n	*pseudo–divisors* < 1000 of n
1	1, 10, 100
2	1, 2, 10, 20, 100, 200

Continued on next page

n	*pseudo–divisors* < 1000 of n
3	1 ,3, 10, 30, 100, 300
4	1, 2, 4, 10, 20, 40, 100, 200, 400
5	1, 5, 10, 50, 100, 500
6	1, 2, 3, 6, 10, 20, 30, 60, 100, 200, 300, 600
7	1, 7, 10, 70, 100, 700
8	1, 2, 4, 8, 10, 20, 40, 80, 100, 200, 400, 800
9	1, 3, 9, 10, 30, 90, 100, 300, 900
10	1, 2, 5, 10, 20, 50, 100, 200, 500
11	1, 11, 101, 110
12	1, 2, 3, 4, 6, 10, 12, 20, 30, 40, 60, 100, 120, 200, 300, 400, 600

4.14.2 Pseudo–Divisors of Second Kind

Definition 4.46. A non–divisor of n is a *pseudo–divisor of second kind* of n if exist a permutation of the digits is a divisor of n.

Table 4.3: Pseudo–divisor of second kind of $n \leq 12$

n	*pseudo–divisors* < 1000 of n
1	10, 100
2	10, 20, 100, 200
3	10, 30, 100, 300
4	10, 20, 40, 100, 200, 400
5	10, 50, 100, 500
6	10, 20, 30, 60, 100, 200, 300, 600
7	10, 70, 100, 700
8	10, 20, 40, 80, 100, 200, 400, 800
9	10, 30, 90, 100, 300, 900
10	10, 20, 50, 100, 200, 500
11	101, 110
12	10, 20, 30, 40, 60, 100, 120, 200, 300, 400, 600

4.14.3 Pseudo–Divisors of Third Kind

Definition 4.47. A number is a *pseudo–divisor of third kind* of n if exist a non-trivial permutation of the digits is a divisor of n.

Table 4.4: Pseudo–divisor of third kind of $n \leq 12$

n	*pseudo–divisors* < 1000 of n
1	10, 100
2	10, 20, 100, 200
3	10, 30, 100, 300
4	10, 20, 40, 100, 200, 400
5	10, 50, 100, 500
6	10, 20, 30, 60, 100, 200, 300, 600
7	10, 70, 100, 700
8	10, 20, 40, 80, 100, 200, 400, 800
9	10, 30, 90, 100, 300, 900
10	10, 20, 50, 100, 200, 500
11	101, 110
12	10, 20, 30, 40, 60, 100, 120, 200, 300, 400, 600

4.15 Pseudo–Odd Numbers

Program 4.48. of counting the odd numbers obtained by digits permutation of the number.

$$
\begin{aligned}
Po(n,i) := \; & m \leftarrow nrd(n,10) \\
& d \leftarrow dn(n,10) \\
& np \leftarrow 1 \;\; if \;\; m{=}1 \\
& np \leftarrow cols(Per2) \;\; if \;\; m{=}2 \\
& np \leftarrow cols(Per3) \;\; if \;\; m{=}3 \\
& np \leftarrow cols(Per4) \;\; if \;\; m{=}4 \\
& sw \leftarrow 0 \\
& for \; j \in i..np \\
& \quad for \; k \in 1..m \\
& \quad\quad pd \leftarrow d \;\; if \; m{=}1 \\
& \quad\quad pd_k \leftarrow d_{(Per2_{k,j})} \;\; if \;\; m{=}2 \\
& \quad\quad pd_k \leftarrow d_{(Per3_{k,j})} \;\; if \;\; m{=}3
\end{aligned}
$$

$$
\begin{array}{l}
\quad\quad pd_k \leftarrow d_{(Per4_{k,j})} \ \ if \ m{=}4 \\
\quad nn \leftarrow pd \cdot Vb(10,m) \\
\quad sw \leftarrow sw+1 \ \ if \ \ \mod(nn,2){=}1 \\
return \ \ sw
\end{array}
$$

The program uses the matrices *Per2* (4.1), *Per3* (4.2) and *Per4* (4.3) which contains all the permutation of the sets $set1,2$, $\{1,2,3\}$ and $\{1,2,3,4\}$.

4.15.1 Pseudo–Odd Numbers of First Kind

Definition 4.49. A number is a *pseudo–odd of first kind* if exist a permutation of digits is an odd number.

Program 4.50. of displaying the *pseudo–odd of first kind.*

$$
APo1(a,b) := \left|
\begin{array}{l}
j \leftarrow 1 \\
for \ n \in a..b \\
\quad if \ Po(n,1) \geq 1 \\
\quad\quad po_j \leftarrow n \\
\quad\quad j \leftarrow j+1 \\
return \ \ po
\end{array}
\right.
$$

This program calls the program *Po*, 4.48.

Pseudo–odd numbers of first kind up to 199 are 175: 1, 3, 5, 7, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 21, 23, 25, 27, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 41, 43, 45, 47, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 61, 63, 65, 67, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 81, 83, 85, 87, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100, 101, 102, 103, 104, 105, 106, 107, 108, 109, 110, 111, 112, 113, 114, 115, 116, 117, 118, 119, 120, 121, 122, 123, 124, 125, 126, 127, 128, 129, 130, 131, 132, 133, 134, 135, 136, 137, 138, 139, 140, 141, 142, 143, 144, 145, 146, 147, 148, 149, 150, 151, 152, 153, 154, 155, 156, 157, 158, 159, 160, 161, 162, 163, 164, 165, 166, 167, 168, 169, 170, 171, 172, 173, 174, 175, 176, 177, 178, 179, 180, 181, 182, 183, 184, 185, 186, 187, 188, 189, 190, 191, 192, 193, 194, 195, 196, 197, 198, 199 .

4.15.2 Pseudo–Odd Numbers of Second Kind

Definition 4.51. Even numbers such that exist a permutation of digits is an odd number.

Program 4.52. of displaying the *pseudo–odd of second kind.*

$$APo2(a,b) := \left| \begin{array}{l} j \leftarrow 1 \\ for\ n \in a..b \\ \quad if\ \ Po(n,1) \geq 1 \wedge \mod(n,2){=}0 \\ \quad \left| \begin{array}{l} po_j \leftarrow n \\ j \leftarrow j+1 \end{array} \right. \\ return\ po \end{array} \right.$$

This program calls the program *Po*, 4.48.

Pseudo–odd numbers of second kind up to 199 are 75: 10, 12, 14, 16, 18, 30, 32, 34, 36, 38, 50, 52, 54, 56, 58, 70, 72, 74, 76, 78, 90, 92, 94, 96, 98, 100, 102, 104, 106, 108, 110, 112, 114, 116, 118, 120, 122, 124, 126, 128, 130, 132, 134, 136, 138, 140, 142, 144, 146, 148, 150, 152, 154, 156, 158, 160, 162, 164, 166, 168, 170, 172, 174, 176, 178, 180, 182, 184, 186, 188, 190, 192, 194, 196, 198 .

4.15.3 Pseudo–Odd Numbers of Third Kind

Definition 4.53. A number is a *pseudo–odd of third kind* if exist a nontrivial permutation of digits is an odd.

Program 4.54. of displaying the *pseudo–odd of third kind*.

$$APo3(a,b) := \left| \begin{array}{l} j \leftarrow 1 \\ for\ n \in a..b \\ \quad if\ \ Po(n,2) \geq 1 \wedge n > 9 \\ \quad \left| \begin{array}{l} po_j \leftarrow n \\ j \leftarrow j+1 \end{array} \right. \\ return\ \ po \end{array} \right.$$

This program calls the program *Po*, 4.48.

Pseudo–odd numbers of third kind up to 199 are 150: 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100, 101, 102, 103, 104, 105, 106, 107, 108, 109, 110, 111, 112, 113, 114, 115, 116, 117, 118, 119, 120, 121, 122, 123, 124, 125, 126, 127, 128, 129, 130, 131, 132, 133, 134, 135, 136, 137, 138, 139, 140, 141, 142, 143, 144, 145, 146, 147, 148, 149, 150, 151, 152, 153, 154, 155, 156, 157, 158, 159, 160, 161, 162, 163, 164, 165, 166, 167, 168, 169, 170, 171, 172, 173, 174, 175, 176, 177, 178, 179, 180, 181, 182, 183, 184, 185, 186, 187, 188, 189, 190, 191, 192, 193, 194, 195, 196, 197, 198, 199 .

4.16 Pseudo–Triangular Numbers

A triangular number has the general form $n(n+1)/2$. The list first 44 triangular numbers is: t^{T} =(1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91, 105, 120, 136, 153, 171, 190, 210, 231, 253, 276, 300, 325, 351, 378, 406, 435, 465, 496, 528, 561, 595, 630, 666, 703, 741, 780, 820, 861, 903, 946, 990) .

Program 4.55. for determining if n is a triangular number or not.

```
IT(n) := | for k ∈ 1..last(t)
         |   | return 0  if t_k > n
         |   | return 1  if t_k = n
         | return 0
```

Program 4.56. for counting the triangular numbers obtained by digits permutation of the number.

```
PT(n, i) := | m ← nrd(n, 10)
            | d ← dn(n, 10)
            | np ← 1 if m=1
            | np ← cols(Per2)  if m=2
            | np ← cols(Per3)  if m=3
            | np ← cols(Per4)  if m=4
            | sw ← 0
            | for j ∈ i..np
            |   | for k ∈ 1..m
            |   |   | pd ← d if m=1
            |   |   | pd_k ← d_(Per2_{k,j})  if m=2
            |   |   | pd_k ← d_(Per3_{k,j})  if m=3
            |   |   | pd_k ← d_(Per4_{k,j})  if m=4
            |   | nn ← pd · Vb(10, m)
            |   | sw ← sw + 1  if IT(nn)=1
            | return sw
```

4.16.1 Pseudo–Triangular Numbers of First Kind

Definition 4.57. A number is a *pseudo–triangular of first kind* if exist a permutation of digits is a triangular number.

Program 4.58. for displaying the *pseudo–triangular of first kind*.

$$APT1(a,b) := \left| \begin{array}{l} j \leftarrow 1 \\ for\ n \in a..b \\ \quad if\ \ PT(n,1) \geq 1 \\ \quad \left| \begin{array}{l} pt_j \leftarrow n \\ j \leftarrow j+1 \end{array} \right. \\ return\ \ pt \end{array} \right.$$

The program calls the program *PT*, 4.56.

Pseudo–triangular numbers of first kind up to 999 are 156: 1, 3, 6, 10, 12, 15, 19, 21, 28, 30, 36, 45, 51, 54, 55, 60, 63, 66, 78, 82, 87, 91, 100, 102, 105, 109, 117, 120, 123, 132, 135, 136, 147, 150, 153, 156, 163, 165, 168, 171, 174, 186, 190, 201, 208, 210, 213, 231, 235, 253, 258, 267, 276, 280, 285, 300, 306, 307, 309, 312, 315, 316, 321, 325, 345, 351, 352, 354, 360, 361, 370, 378, 387, 390, 405, 406, 417, 435, 450, 453, 456, 460, 465, 469, 471, 496, 501, 504, 505, 510, 513, 516, 523, 528, 531, 532, 534, 540, 543, 546, 550, 559, 561, 564, 582, 595, 600, 603, 604, 606, 613, 615, 618, 627, 630, 631, 640, 645, 649, 651, 654, 660, 666, 672, 681, 694, 703, 708, 711, 714, 726, 730, 738, 741, 762, 780, 783, 802, 807, 816, 820, 825, 837, 852, 861, 870, 873, 901, 903, 909, 910, 930, 946, 955, 964, 990 . This numbers obtain with the command $APT1(1,999)^{\mathrm{T}} =$, where *APT1* is the program 4.58.

4.16.2 Pseudo–Triangular Numbers of Second Kind

Definition 4.59. A non–triangular number is a *pseudo–triangular of second kind* if exist a permutation of the digits is a triangular number.

Program 4.60. for displaying the *pseudo–triangular of second kind.*

$$APT2(a,b) := \left| \begin{array}{l} j \leftarrow 1 \\ for\ n \in a..b \\ \quad if\ \ IT(n){=}0 \wedge PT(n,1) \geq 1 \\ \quad \left| \begin{array}{l} pt_j \leftarrow n \\ j \leftarrow j+1 \end{array} \right. \\ return\ \ pt \end{array} \right.$$

The program calls the programs *IT*, 4.55 and *PT*, 4.56.

Pseudo–triangular numbers of second kind up to 999 are 112: 12, 19, 30, 51, 54, 60, 63, 82, 87, 100, 102, 109, 117, 123, 132, 135, 147, 150, 156, 163, 165, 168, 174, 186, 201, 208, 213, 235, 258, 267, 280, 285, 306, 307, 309, 312, 315, 316, 321, 345, 352, 354, 360, 361, 370, 387, 390, 405, 417, 450, 453, 456, 460, 469, 471, 501, 504, 505, 510, 513, 516, 523, 531, 532, 534, 540, 543, 546, 550, 559, 564, 582, 600, 603, 604, 606, 613, 615, 618, 627, 631, 640, 645, 649, 651, 654, 660, 672, 681, 694, 708, 711, 714, 726, 730, 738, 762, 783, 802, 807, 816, 825, 837, 852, 870, 873,

901, 909, 910, 930, 955, 964 . This numbers are obtained with the command $APT2(1,999)^{\mathrm{T}} =$, where *APT2* is the program 4.60.

4.16.3 Pseudo–Triangular Numbers of Third Kind

Definition 4.61. A number is a *pseudo–triangular of third kind* if exist a non-trivial permutation of the digits is a triangular number.

Program 4.62. for displaying the *pseudo–triangular of third kind*.

```
APT3(a,b) := | j ← 1
             | for n ∈ a..b
             |   if  PT(n,2) ≥ 1 ∧ n > 9
             |     | pt_j ← n
             |     | j ← j + 1
             | return  pt
```

The program calls the program *PT*, 4.56.

Pseudo–triangular numbers of third kind up to 999 are 133: 10, 12, 19, 30, 51, 54, 55, 60, 63, 66, 82, 87, 100, 102, 105, 109, 117, 120, 123, 132, 135, 147, 150, 153, 156, 163, 165, 168, 171, 174, 186, 190, 201, 208, 210, 213, 235, 253, 258, 267, 280, 285, 300, 306, 307, 309, 312, 315, 316, 321, 325, 345, 351, 352, 354, 360, 361, 370, 387, 390, 405, 417, 450, 453, 456, 460, 469, 471, 496, 501, 504, 505, 510, 513, 516, 523, 531, 532, 534, 540, 543, 546, 550, 559, 564, 582, 595, 600, 603, 604, 606, 613, 615, 618, 627, 630, 631, 640, 645, 649, 651, 654, 660, 666, 672, 681, 694, 708, 711, 714, 726, 730, 738, 762, 780, 783, 802, 807, 816, 820, 825, 837, 852, 870, 873, 901, 909, 910, 930, 946, 955, 964, 990 . This numbers are obtained with the command $APT3(1,999)^{\mathrm{T}} =$, where *APT3* is the program 4.62.

4.17 Pseudo–Even Numbers

With similar programs with programs *Po*, 4.48, *Apo1*, 4.50, *APo2*, 4.52 and *APo3*, 4.54 can get *pseudo–even numbers*.

4.17.1 Pseudo–even Numbers of First Kind

Definition 4.63. A number is a *pseudo–even of first kind* if exist a permutation of digits is an even number.

Pseudo–even numbers of first kind up to 199 are 144: 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 32, 34, 36, 38, 40, 41, 42, 43, 44, 45, 46,

47, 48, 49, 50, 52, 54, 56, 58, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 72, 74, 76, 78, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 92, 94, 96, 98, 100, 101, 102, 103, 104, 105, 106, 107, 108, 109, 110, 112, 114, 116, 118, 120, 121, 122, 123, 124, 125, 126, 127, 128, 129, 130, 132, 134, 136, 138, 140, 141, 142, 143, 144, 145, 146, 147, 148, 149, 150, 152, 154, 156, 158, 160, 161, 162, 163, 164, 165, 166, 167, 168, 169, 170, 172, 174, 176, 178, 180, 181, 182, 183, 184, 185, 186, 187, 188, 189, 190, 192, 194, 196, 198.

4.17.2 Pseudo–Even Numbers of Second Kind

Definition 4.64. Odd numbers such that exist a permutation of digits is an even number.

Pseudo–even numbers of second kind up to 199 are 45: 21, 23, 25, 27, 29, 41, 43, 45, 47, 49, 61, 63, 65, 67, 69, 81, 83, 85, 87, 89, 101, 103, 105, 107, 109, 121, 123, 125, 127, 129, 141, 143, 145, 147, 149, 161, 163, 165, 167, 169, 181, 183, 185, 187, 189.

4.17.3 Pseudo–Even Numbers of Third Kind

Definition 4.65. A number is a *pseudo–even of third kind* if exist a nontrivial permutation of digits is an even.

Pseudo–even numbers of third kind up to 199 are 115: 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 100, 101, 102, 103, 104, 105, 106, 107, 108, 109, 110, 112, 114, 116, 118, 120, 121, 122, 123, 124, 125, 126, 127, 128, 129, 130, 132, 134, 136, 138, 140, 141, 142, 143, 144, 145, 146, 147, 148, 149, 150, 152, 154, 156, 158, 160, 161, 162, 163, 164, 165, 166, 167, 168, 169, 170, 172, 174, 176, 178, 180, 181, 182, 183, 184, 185, 186, 187, 188, 189, 190, 192, 194, 196, 198.

4.18 Pseudo–Multiples of Prime

4.18.1 Pseudo–Multiples of First Kind of Prime

Definition 4.66. A number is a *pseudo–multiple of first kind* of *prime* if exist a permutation of the digits is a multiple of p, including the identity permutation.

Pseudo–Multiples of first kind of 5 up to 199 are 63: 5, 10, 15, 20, 25, 30, 35, 40, 45, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 65, 70, 75, 80, 85, 90, 95, 100, 101, 102, 103, 104, 105, 106, 107, 108, 109, 110, 115, 120, 125, 130, 135, 140, 145, 150, 151, 152, 153, 154, 155, 156, 157, 158, 159, 160, 165, 170, 175, 180, 185, 190, 195.

Pseudo–Multiples of first kind of 7 up to 199 are 80: 7, 12, 14, 19, 21, 24, 28, 35, 36, 41, 42, 48, 49, 53, 56, 63, 65, 70, 77, 82, 84, 89, 91, 94, 98, 102, 103, 104, 105, 109, 112, 115, 116, 119, 120, 121, 123, 126, 127, 128, 130, 132, 133, 134, 135, 137, 139, 140, 143, 144, 145, 147, 150, 151, 153, 154, 156, 157, 158, 161, 162, 165, 166, 168, 169, 172, 173, 174, 175, 179, 182, 185, 186, 189, 190, 191, 193, 196, 197, 198.

4.18.2 Pseudo–Multiples of Second Kind of Prime

Definition 4.67. A non-multiple of p is a *pseudo–multiple of second kind* of p (prime) if exist permutation of the digits is a multiple of p.

Pseudo–Multiples of second kind of 5 up to 199 are 24: 51, 52, 53, 54, 56, 57, 58, 59, 101, 102, 103, 104, 106, 107, 108, 109, 151, 152, 153, 154, 156, 157, 158, 159.

Pseudo–Multiples of second kind of 7 up to 199 are 52: 12, 19, 24, 36, 41, 48, 53, 65, 82, 89, 94, 102, 103, 104, 109, 115, 116, 120, 121, 123, 127, 128, 130, 132, 134, 135, 137, 139, 143, 144, 145, 150, 151, 153, 156, 157, 158, 162, 165, 166, 169, 172, 173, 174, 179, 185, 186, 190, 191, 193, 197, 198.

4.18.3 Pseudo–Multiples of Third Kind of Prime

Definition 4.68. A number is a *pseudo–multiple of third kind* of p (prime) if exist a nontrivial permutation of the digits is a multiple of p.

Pseudo–Multiples of third kind of 5 up to 199 are 46: 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 100, 101, 102, 103, 104, 105, 106, 107, 108, 109, 110, 115, 120, 125, 130, 135, 140, 145, 150, 151, 152, 153, 154, 155, 156, 157, 158, 159, 160, 165, 170, 175, 180, 185, 190, 195.

Pseudo–Multiples of third kind of 7 up to 199 are 63: 12, 19, 24, 36, 41, 48, 53, 65, 70, 77, 82, 89, 94, 102, 103, 104, 109, 112, 115, 116, 119, 120, 121, 123, 127, 128, 130, 132, 133, 134, 135, 137, 139, 140, 143, 144, 145, 147, 150, 151, 153, 156, 157, 158, 161, 162, 165, 166, 168, 169, 172, 173, 174, 179, 182, 185, 186, 189, 190, 191, 193, 197, 198.

4.19 Progressions

How many primes do the following progressions contain:

1. The sequence $\{a \cdot p_n + b\}$, $n = 1, 2, \ldots$ where $(a, b) = 1$, i.e. *gcd(a,b)* = 1, and p_n is n-th prime?

Example: $a := 3$ $b := 10$ $n := 25$ $k := 1..n$ $q_k := a \cdot p_k + b$, then $q^{\mathrm{T}} \rightarrow (2^4,$ $19, 5^2, 31, 43, 7^2, 61, 67, 79, 97, 103, 11^2, 7 \cdot 19, 139, 151, 13^2, 11 \cdot 17, 193,$ $211, 223, 229, 13 \cdot 19, 7 \cdot 37, 277, 7 \cdot 43)$. Therefore in 25 terms 15 are prime numbers (See Figure 10.2).

2. The sequence $\{a^n + b\}$, $n = 1, 2, \ldots$, where $(a, b) = 1$, and $a \neq \pm 1$ and $a \neq 0$?

   Example: $a := 3$ $b := 10$ $n := 25$ $k := 1..n$ $q_k := a^k + b$, then in sequence q are 6 prime numbers: 13, 19, 37, 739, 65571 and 387420499 (See 10.2).

3. The sequence $\{n^n \pm 1\}$, $n = 1, 2, \ldots$?

   (a) First 10 terms from the sequence $\{n^n + 1\}$ are: 2, 5, 28, 257, 3126, 46657, 823544, 16777217, 387420490, 10000000001, of which 2, 5 and 257 are primes (See Figure 10.2).

   (b) First 10 terms from the sequence $\{n^n - 1\}$ are: 0, 3, 26, 255, 3124, 46655, 823542, 16777215, 387420488, 9999999999, of which 3 is prime (See Figure 10.2).

4. The sequence $\{p_n\# \pm 1\}$, $n = 1, 2, \ldots$, where p_n is n-th prime?

   (a) First 10 terms from the sequence $\{p_n\# + 1\}$ are: 3, 7, 31, 211, 2311, 30031, 510511, 9699691, 223092871, 6469693231 of which 3, 7, 31 and 211 are primes (See Figure 10.2).

   (b) First 10 terms from the sequence $\{p_n\# - 1\}$ are: 1, 5, 29, 209, 2309, 30029, 510509, 9699689, 223092869, 6469693229 of which 5, 29, 2309 and 30029 are primes (See Figure 10.2).

5. The sequence $\{p_n\#\# \pm 1\}$, $n = 1, 2, \ldots$, where p_n is n-th prime?

   (a) First 10 terms from the sequence $\{p_n\#\# + 1\}$ are: 3, 4, 11, 8, 111, 92, 1871, 1730, 43011, 1247291 of which 3, 11, 1871 and 1247291 are primes (See Figure 10.2).

   (b) First 10 terms from the sequence $\{p_n\#\# - 1\}$ are: 1, 2, 9, 6, 109, 90, 1869, 1728, 43009, 1247289 of which 2 and 109 are primes (See Figure 10.2).

6. The sequence $\{p_n\#\#\# \pm 2\}$, $n = 1, 2, \ldots$, where p_n is n-th prime?

   (a) First 10 terms from the sequence $\{p_n\#\#\# + 2\}$ are: 4, 5, 7, 23, 233, 67, 1107, 4391, 100949, 32047 of which 5, 7, 23, 233, 67 and 4391 are primes (See Figure 10.2).

(b) First 10 terms from the sequence $\{p_n\#\#\#-2\}$ are: 0, 1, 3, 19, 229, 63, 1103, 4387, 100945, 32043 of which 3, 19, 229 and 1103 are primes (See Figure 10.2).

7. The sequence $\{n!!\pm 2\}$ and $\{n!!!\pm 1\}$, $n = 1, 2, \ldots$?

   (a) First 17 terms from the sequence $\{n!!+2\}$ are: 3, 4, 5, 10, 17, 50, 107, 386, 947, 3842, 10397, 46082, 135137, 645122, 2027027, 10321922, 34459427 of which 3, 5, 17, 107 and 947 are primes (See Figure 10.2).

   (b) First 17 terms from the sequence $\{n!!-2\}$ are: -1, 0, 1, 6, 13, 46, 103, 382, 943, 3838, 10393, 46078, 135133, 645118, 2027023, 10321918, 34459423 of which 13, 103, 2027023 and 34459423 are primes (See Figure 10.2).

   (c) First 22 terms from the sequence $\{n!!!+1\}$ are: 2, 3, 4, 5, 11, 19, 29, 81, 163, 281, 881, 1945, 3641, 12321, 29161, 58241, 209441, 524881, 1106561, 4188801, 11022481, 24344321 of which 2, 3, 5, 11, 19, 29, 163, 281, 881, and 209441 are primes (See Figure 10.2).

   (d) First 22 terms from the sequence $\{n!!!-1\}$ are: 0, 1, 2, 3, 9, 17, 27, 79, 161, 279, 879, 1943, 3639, 12319, 29159, 58239, 209439, 524879, 1106559, 4188799, 11022479, 24344319 of which 2, 3, 17, 79 and 4188799 are primes (See Figure 10.2).

8. The sequences $\{2^n \pm 1\}$ (Mersenne primes) and $\{n!\pm 1\}$ (factorial primes) are well studied.

4.20 Palindromes

4.20.1 Classical Palindromes

A palindrome of one digit is a number (in some base b) that is the same when written forwards or backwards, i.e. of the form $\overline{d_1 d_2 \ldots d_2 d_1}$. The first few palindrome in base 10 are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 22, 33, 44, 55, 66, 77, 88, 99, 101, 111, 121, ... [Sloane, 2014, A002113].

The numbers of palindromes less than 10, 10^2, 10^3, ... are 9, 18, 108, 198, 1098, 1998, 10998, ... [Sloane, 2014, A050250].

Program 4.69. palindrome generator in base b.

$$gP(v,b) := \left| \begin{array}{l} for\ k \in 1..last(v) \\ \quad \left| \begin{array}{l} n \leftarrow n + v_k \cdot b^m \\ m \leftarrow m + nrd(v_k, b) \end{array} \right. \end{array} \right.$$

```
    return n
```

The program use the function *nrd*, 2.1.

Program 4.70. of generate palindromes and primality verification.

```
PgP(α, β, r, b, t, IsP) := | j ← 0
                           | for k1 ∈ α, α + r..β
                           |   v1 ← k1
                           |   for k2 ∈ α, α + r..β
                           |     v2 ← k2
                           |     for k3 ∈ α, α + r..β
                           |       v3 ← k3
                           |       n ← gP(v,b)
                           |       if IsP = 1
                           |         if IsPrime(n)=1
                           |           j ← j + 1
                           |           S_j ← n
                           |       otherwise
                           |         j ← j + 1
                           |         S_j ← n
                           | return sort(S)
```

This program use subprogram *gP*, 4.69 and Mathcad programs *IsPrime* and *sort*.

There are 125 palindromes of 5–digits, in base 10, made only with numbers 1, 3, 5, 7 and 9, which are obtained by running $Pgp(1,9,2,10,0,0)^{\mathrm{T}} \rightarrow$ 11111, 11311, 11511, 11711, 11911, 13131, 13331, 13531, 13731, 13931, 15151, 15351, 15551, 15751, 15951, 17171, 17371, 17571, 17771, 17971, 19191, 19391, 19591, 19791, 19991, 31113, 31313, 31513, 31713, 31913, 33133, 33333, 33533, 33733, 33933, 35153, 35353, 35553, 35753, 35953, 37173, 37373, 37573, 37773, 37973, 39193, 39393, 39593, 39793, 39993, 51115, 51315, 51515, 51715, 51915, 53135, 53335, 53535, 53735, 53935, 55155, 55355, 55555, 55755, 55955, 57175, 57375, 57575, 57775, 57975, 59195, 59395, 59595, 59795, 59995, 71117, 71317, 71517, 71717, 71917, 73137, 73337, 73537, 73737, 73937, 75157, 75357, 75557, 75757, 75957, 77177, 77377, 77577, 77777, 77977, 79197, 79397, 79597, 79797, 79997, 91119, 91319, 91519, 91719, 91919, 93139, 93339, 93539, 93739, 93939, 95159, 95359, 95559, 95759, 95959, 97179, 97379, 97579, 97779, 97979, 99199, 99399, 99599, 99799, 99999.

Of these we have 25 prime numbers, which are obtained by running $Pgp(1,9,2,10,0,1)^{\mathrm{T}} \rightarrow$ 11311, 13331, 13931, 15551, 17971, 19391, 19991, 31513,

33533, 35153, 35353, 35753, 37573, 71317, 71917, 75557, 77377, 77977, 79397, 79997, 93139, 93739, 95959, 97379, 97579 .

Program 4.71. the palindromes recognition in base b.

```
RePal(n,b) := | d ← dn(n,b)
              | u ← last(d)
              | return 1 if n=1
              | m ← floor(u/2)
              | for k ∈ 1..m
              |    return 0  if  dk ≠ d_{u−k+1}
              | return 1
```

Program 4.72. the palindromes counting.

```
NrPa(m,B) := | for b ∈ 2..B
             |  | for k ∈ 1..b^m
             |  |    v_k ← RePal(k,b)
             |  | for μ ∈ 1..m
             |  |    NP_{b−1,μ} ← ∑ submatrix(v,1,b^μ,1,1)
             | return NP
```

The number of palindromes of one digit in base b is given in Table 4.5 and was obtained with the command *NrPa*(6,16).

4.20.2 Palindromes with Groups of m Digits

1. Palindromes with groups of one digit in base b are classical palindromes.

2. Palindromes with groups of 2 digits, in base b, are:

$$\overline{d_1d_2d_3d_4\ldots d_{n-3}d_{n-2}d_{n-1}d_nd_{n-1}d_nd_{n-3}d_{n-2}\ldots d_3d_4d_1d_2}$$

or

$$\overline{d_1d_2d_3d_4\ldots d_{n-3}d_{n-2}d_{n-1}d_nd_{n-3}d_{n-2}\ldots d_3d_4d_1d_2}\ ,$$

where $d_k \in \{0,1,2,\ldots,b-1\}$ and $b \in \mathbb{N}^*$, $b \geq 2$.

Examples: 345534, 78232378, 782378, 105565655510, 1055655510, 3334353636353433, 33343536353433.

$b\backslash b^k$	b	b^2	b^3	b^4	b^5	b^6
2	1	2	4	6	10	14
3	2	4	10	16	34	52
4	3	6	18	30	78	126
5	4	8	29	49	149	250
6	5	10	41	71	251	432
7	6	12	54	96	390	684
8	7	14	70	126	574	1022
9	8	16	88	160	808	1456
10	9	18	108	198	1098	1998
11	10	20	130	240	1451	2661
12	11	22	154	286	1871	3455
13	12	24	180	336	2364	4392
14	13	26	208	390	2938	5486
15	14	28	239	449	3600	6751
16	15	30	270	510	4350	8190

Table 4.5: Number of palindromes of one digit in base b

3. Palindromes with groups of 3 digits in base b, are:

$$\overline{d_1d_2d_3d_4d_5d_6\ldots d_{n-2}d_{n-1}d_nd_{n-2}nd_{n-1}d_n\ldots d_4d_5d_6d_1d_2d_3}$$

   or

$$\overline{d_1d_2d_3\ldots d_{n-5}d_{n-4}d_{n-3}d_{n-2}d_{n-1}d_nd_{n-5}d_{n-4}d_{n-3}\ldots d_1d_2d_3}$$

   where $d_k \in \{0,1,2,\ldots,b-1\}$ and $b \in \mathbb{N}^*$, $b \geq 2$. Examples: 987987, 456567678678567456, 456567678567456, 123321123, 123234234123, 676767808808767676.

4. and so on .

Examples of palindromes with groups of 2 digits in base $b = 3$ are: 1, 2, 30 = $1010_{(3)}$, 40 = $1111_{(3)}$, 50 = $1212_{(3)}$, 60 = $2020_{(3)}$, 70 = $2121_{(3)}$, 80 = $2222_{(3)}$, or in base $b = 4$ are: 1, 2, 3, 68 = $1010_{(4)}$, 85 = $1111_{(4)}$, 102 = 1212 $=_{(4)}$, 119 = $1313_{(4)}$, 136 = $2020_{(4)}$, 153 = $2121_{(4)}$, 170 = $2222_{(4)}$, 187 = $2323_{(4)}$, 204 = $3030_{(4)}$, 221 = $3131_{(4)}$, 238 = $3232_{(4)}$, 255 = $3333_{(4)}$. It is noted that $1111_{(3)}$, $2222_{(3)}$, $1111_{(4)}$, $2222_{(4)}$ and $3333_{(4)}$ are palindromes and a single digit.

Numbers of palindromes with groups of one and two digits, in base b, $b = 2,3,\ldots,16$, for the numbers $1,2,\ldots,b^m$, where $m = 1,2,\ldots,6$ are found in Table 4.6.

$b\backslash b^k$	b	b^2	b^3	b^4	b^5	b^6
2	1	2	4	7	13	23
3	2	4	10	20	50	116
4	3	6	18	39	123	351
5	4	8	29	65	245	826
6	5	10	41	96	426	1657
7	6	12	54	132	678	2988
8	7	14	70	175	1015	4991
9	8	16	88	224	1448	7856
10	9	18	108	279	1989	11799
11	10	20	130	340	2651	17061
12	11	22	154	407	3444	23904
13	12	24	180	480	4380	32616
14	13	26	208	559	5473	43511
15	14	28	239	645	6736	56927
16	15	30	270	735	8175	73215

Table 4.6: Number of palindromes of one and two digits in base b

Numbers of palindromes with groups of one, two and three digits, in base $b=2,3,\ldots,16$, for the numbers $1,2,\ldots,b^m$ where $m=1,2,\ldots,6$ are found in Table 4.7.

Unsolved research problem: The interested readers can study the m-digits palindromes that are prime, considering special classes of m-digits palindromes.

4.20.3 Generalized Smarandache Palindrome

A generalized Smarandache palindrome (GSP) is a number of the concatenated form:

$$\overline{a_1 a_2 \ldots a_{n-1} a_n a_{n-1} \ldots a_2 a_1}$$

with $n \geq 2$ (GSP1), or

$$\overline{a_1 a_2 \ldots a_{n-1} a_n a_n a_{n-1} \ldots a_2 a_1}$$

with $n \geq 1$ (GSP2), where all a_1, a_2, $\ldots$, a_n are positive integers in base b of various number of digits, [Khoshnevisan, 2003a,b, Evans et al., 2004], [Sloane, 2014, A082461], [Weisstein, 2015b,c].

We agree that any number with a single digit, in base of numeration b is palindrome GSP1 and palindrome GSP2.

Examples:

$b\backslash b^k$	b	b^2	b^3	b^4	b^5	b^6
2	1	2	4	7	13	25
3	2	4	10	20	50	128
4	3	6	18	39	123	387
5	4	8	29	65	245	906
6	5	10	41	96	426	1807
7	6	12	54	132	678	3240
8	7	14	70	175	1015	5383
9	8	16	88	224	1448	8432
10	9	18	108	279	1989	12609
11	10	20	130	340	2651	18161
12	11	22	154	407	3444	25356
13	12	24	180	480	4380	34488
14	13	26	208	559	5473	45877
15	14	28	239	645	6736	59867
16	15	30	270	735	8175	76815

Table 4.7: Number of palindromes of one, two and three digits in base b

1. The number $123567567312_{(10)}$ is a GSP2 because we can group it as (12)(3)(567)(567)(3)(12) i.e. ABCCBA.
2. The number $23523_{(8)} = 10067_{(10)}$ is also a GSP1 since we can group it as (23)(5)(23), i.e. ABA.
3. The number $abcddcba_{(16)} = 2882395322_{(10)}$ is a GSP2.

Program 4.73.

$$GSP1(v,b) := gP(stack(v, submatrix(reverse(v), 2, last(v), 1, 1)), b)\ ,$$

where *stack*, *submatrix* and *reverse* are Mathcad functions.

Program 4.74.

$$GSP2(v,b) := gP(stack(v, reverse(v)), b)\ ,$$

where *stack* and *reverse* are Mathcad functions.

Examples:

1. If $v := \begin{pmatrix} 17 & 3 & 567 \end{pmatrix}^{\mathrm{T}}$, then
   (a) $GSP1(v,10) = 173567317_{(10)}$ and $GSP2(v,10) = 173567567317_{(10)}$,
   (b) $GSP1(v,8) = 291794641_{(10)} = 173567317_{(8)}$ and
   $GSP2(v,8) = 1195190283985_{(10)} = 173567567317_{(8)}$.

2. If $u := \begin{pmatrix} 31 & 3 & 201 & 1013 \end{pmatrix}^{\mathrm{T}}$ then

   (a) $GSP1(u,10) = 3132011013201331_{(10)}$ and
   $GSP2(u,10) = 31320110131013201331_{(10)}$,

   (b)

$$GSP1(u,5) = 120790190751031_{(10)} = 3132011013201331_{(5)} \text{ and}$$
$$GSP2(u,5) = 31320110131013201331_{(10)}$$
$$= 31320110131013201331_{(5)} .$$

Program 4.75. for generating the palindrome in base b of type GSP1 or GSP2 and eventually checking the primality.

```
PgGSP(α, β, ρ, b, f, IsP) := | j ← 0
                             | for k1 ∈ α, α + ρ..β
                             |   | v1 ← k1
                             |   | for k2 ∈ α, α + ρ..β
                             |   |   | v2 ← k2
                             |   |   | for k3 ∈ α, α + ρ..β
                             |   |   |   | v3 ← k3
                             |   |   |   | n ← f(n, b)
                             |   |   |   | if IsPrime(n)=1  if IsP = 1
                             |   |   |   |   | j ← j + 1
                             |   |   |   |   | S_j ← n
                             |   |   |   | otherwise
                             |   |   |   |   | j ← j + 1
                             |   |   |   |   | S_j ← n
                             | return sort(S)
```

Examples:

1. All palindromes, in base of numeration $b = 10$, of 5 numbers from the set $\{1,3,5,7,9\}$ is obtained with the command

$$z = PgGSP(1,9,2,10,GSP1,0)$$

   and to display the vector z: $z^{\mathrm{T}} \to$ 11111, 11311, 11511, 11711, 11911, 13131, 13331, 13531, 13731, 13931, 15151, 15351, 15551, 15751, 15951, 17171, 17371, 17571, 17771, 17971, 19191, 19391, 19591, 19791, 19991, 31113, 31313, 31513, 31713, 31913, 33133, 33333, 33533, 33733, 33933, 35153, 35353, 35553, 35753, 35953, 37173, 37373, 37573, 37773, 37973,

39193, 39393, 39593, 39793, 39993, 51115, 51315, 51515, 51715, 51915, 53135, 53335, 53535, 53735, 53935, 55155, 55355, 55555, 55755, 55955, 57175, 57375, 57575, 57775, 57975, 59195, 59395, 59595, 59795, 59995, 71117, 71317, 71517, 71717, 71917, 73137, 73337, 73537, 73737, 73937, 75157, 75357, 75557, 75757, 75957, 77177, 77377, 77577, 77777, 77977, 79197, 79397, 79597, 79797, 79997, 91119, 91319, 91519, 91719, 91919, 93139, 93339, 93539, 93739, 93939, 95159, 95359, 95559, 95759, 95959, 97179, 97379, 97579, 97779, 97979, 99199, 99399, 99599, 99799, 99999 and $length(z) \to 125$.

2. All prime palindromes, in base $b = 10$, of 5 numbers from the set $\{1,3,5,7,9\}$ is obtained with the command

$$zp = PgGSP(1,9,2,10,GSP1,1)$$

and to display the vector zp: $zp^{\mathrm{T}} \to$ 11311, 13331, 13931, 15551, 17971, 19391, 19991, 31513, 33533, 35153, 35353, 35753, 37573, 71317, 71917, 75557, 77377, 77977, 79397, 79997, 93139, 93739, 95959, 97379, 97579 and $length(zp) \to 25$

3. All prime palindromes, in base $b = 10$, of 5 numbers from the set $\{1,4,7,10,13\}$ is obtained with the command

$$sp = PgGSP(1,13,3,10,GSP1,1)$$

and to display the vector sp: $sp^{\mathrm{T}} \to$ 11411, 14741, 17471, 74747, 77477, 141041, 711017, 711317, 741347, 1104101, 1107101, 1131131, 1314113, 1347413, 1374713, 1377713, 7104107, 7134137, 13410413, 131371313 and $length(sp) \to 20$.

Program 4.76. of recognition *GSP* the number n in base b.

```
RecGSP(n, b) := | d ← dn(n, b)
                | m ← length(d)
                | return 1 if m=1
                | jm ← floor(m/2)
                | for j ∈ 1..jm
                |   | d1 ← submatrix(d, 1, j, 1, 1)
                |   | d2 ← submatrix(d, m+1−j, m, 1, 1)
                |   | return 1 if d1 = d2
                | return 0
```

Program 4.77. of search palindromes GSP ($y = 1$) or not palindromes GSP ($y = 0$) from α to β in base b.

$$PGSP(\alpha,\beta,b,y) := \left| \begin{array}{l} j \leftarrow 0 \\ for\ n \in \alpha..\beta \\ \quad if\ RecGSP(n,b){=}y \\ \quad \left| \begin{array}{l} j \leftarrow j+1 \\ S_{j,1} \leftarrow n \\ S_{j,2} \leftarrow dn(n,b) \end{array} \right. \\ return\ S \end{array} \right.$$

With this program can display palindromes GSP, in base $b = 2$, from 1 by 16:

$$PGSP(1,2^4,2,1) = \begin{bmatrix} 1 & (1) \\ 3 & (1\ \ 1) \\ 5 & (1\ \ 0\ \ 1) \\ 7 & (1\ \ 1\ \ 1) \\ 9 & (1\ \ 0\ \ 0\ \ 1) \\ 10 & (1\ \ 0\ \ 1\ \ 0) \\ 11 & (1\ \ 0\ \ 1\ \ 1) \\ 13 & (1\ \ 1\ \ 0\ \ 1) \\ 15 & (1\ \ 1\ \ 1\ \ 1) \end{bmatrix}.$$

Program 4.78. the GSP palindromes counting.

$$NrGSP(m,B) := \left| \begin{array}{l} for\ b \in 2..B \\ for\ k \in 1..b^m \\ \quad \left| \begin{array}{l} v_k \leftarrow 1\ if\ RecGSP(k,b){=}1 \\ v_k \leftarrow 0\ otherwise \\ for\ \mu \in 1..m \\ \quad NP_{b-1,\mu} \leftarrow \sum submatrix(v,1,b^{\mu},1,1) \\ v \leftarrow 0 \end{array} \right. \\ return\ \ NP \end{array} \right.$$

The number of palindromes GSP, in base b, are given in Table 4.8, using the command $NrGSP(6,16)$:

Unsolved research problem: To study the number of prime GSPs for given classes of GSPs.

$b\backslash b^k$	b	b^2	b^3	b^4	b^5	b^6
2	1	2	4	9	19	41
3	2	4	10	32	98	308
4	3	6	18	75	303	1251
5	4	8	29	145	725	3706
6	5	10	41	246	1476	9007
7	6	12	54	384	2694	19116
8	7	14	70	567	4543	36743
9	8	16	88	800	7208	65456
10	9	18	108	1089	10899	109809
11	10	20	130	1440	15851	175461
12	11	22	154	1859	22320	269292
13	12	24	180	2352	30588	399528
14	13	26	208	2925	40963	575861
15	14	28	239	3585	53776	809567
16	15	30	270	4335	69375	1113615

Table 4.8: Number of palindromes GSP in base b

4.21 Smarandache–Wellin Primes

1. Special prime digital subsequence: 2, 3, 5, 7, 23, 37, 53, 73, 223, 227, 233, 257, 277, 337, 353, 373, 523, 557, 577, 727, 733, 757, 773, 2237, 2273, 2333, 2357, 2377, 2557, 2753, 2777, 3253, 3257, 3323, 3373, 3527, 3533, 3557, 3727, 3733, 5227, 5233, 5237, 5273, 5323, 5333, 5527, 5557, 5573, 5737, 7237, 7253, 7333, 7523, 7537, 7573, 7577, 7723, 7727, 7753, 7757 ..., i.e. the prime numbers whose digits are all primes (they are called *Smarandache–Wellin primes*). For all primes up to 10^7, which are in number 664579, 1903 are *Smarandache–Wellin primes*.

   Conjecture: this sequence is infinite.

   Program 4.79. of generate primes Wellin.

```
Wellin(p,b,L) := | for k ∈ 1..L
                 |   | d ← dn(p_k, b)
                 |   | sw1 ← 0
                 |   | for j ∈ 1..last(d)
                 |   |   | h ← 1
                 |   |   | sw2 ← 0
                 |   |   | while p_h ≤ b
                 |   |   |   | if p_h ≤ b
```

$$
\begin{array}{l}
\quad\quad\quad\quad sw2 \leftarrow 1 \\
\quad\quad\quad\quad break \\
\quad\quad\quad h \leftarrow h+1 \\
\quad\quad sw1 \leftarrow sw1+1 \;\; if \;\; sw2=1 \\
\quad if \;\; sw1 = last(d) \\
\quad\quad i \leftarrow i+1 \\
\quad\quad w_i \leftarrow p_k \\
return \;\; w
\end{array}
$$

The list *Smarandache-Wellin primes* generate with commands $L := 1000$ $p := submatrix(prime, 1, L, 1, 1)$ and $Wellin(p, 10, L) =$.

2. *Cira–Smarandache–Wellin primes in octal base*, are that have digits only primes up to 8, i.e. digits are: 2, 3, 5 and 7. For the first 1000 primes, exist 82 of *Cira–Smarandache–Wellin primes in octal base*: 2, 3, 5, 7, 23, 27, 35, 37, 53, 57, 73, 75, 225, 227, 235, 255, 277, 323, 337, 357, 373, 533, 535, 557, 573, 577, 723, 737, 753, 775, 2223, 2235, 2275, 2325, 2353, 2375, 2377, 2527, 2535, 2725, 2733, 2773, 3235, 3255, 3273, 3323, 3337, 3373, 3375, 3525, 3527, 3555, 3723, 3733, 3753, 3755, 5223, 5227, 5237, 5253, 5275, 5355, 5527, 5535, 5557, 5573, 5735, 5773, 7225, 7233, 7325, 7333, 7355, 7357, 7523, 7533, 7553, 7577, 7723, 7757, 7773, 7775 . Where, for example, $7775_{(8)} = 4093_{(10)}$. The list *Smarandache-Wellin primes in octal base* generate with commands $L := 1000$ $p := submatrix(prime, 1, L, 1, 1)$ and $Wellin(p, 8, L) =$.

3. *Cira–Smarandache–Wellin primes in hexadecimal base*, are that have digits only primes up to 16, i.e. digits are: 2, 3, 5, 7, b and d. For the first 1000 primes, exist 68 of *Cira–Smarandache–Wellin primes in hexadecimal base*: 2, 3, 5, 7, b, d, 25, 2b, 35, 3b, 3d, 53, b3, b5, d3, 223, 22d, 233, 23b, 257, 277, 2b3, 2bd, 2d7, 2dd, 32b, 335, 337, 33b, 33d, 355, 35b, 373, 377, 3b3, 3d7, 527, 557, 55d, 577, 5b3, 5b5, 5db, 727, 737, 755, 757, 773, 7b5, 7bb, 7d3, 7db, b23, b2d, b57, b5d, b7b, bb7, bdd, d2b, d2d, d3d, d55, db7, dbd, dd3, dd5, ddb. Where, for example, $ddb_{(16)} = 3547_{(10)}$. The list *Smarandache-Wellin primes in hexadecimal base* generate with commands $L := 1000$ $p := submatrix(prime, 1, L, 1, 1)$ and $Wellin(p, 16, L) =$.

4. The primes that have numbers of 2 digits primes are *Cira–Smarandache–Wellin primes of second order*. The list the *Cira–Smarandache–Wellin primes of second order*, from 1000 primes, is: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 211, 223, 229, 241, 271, 283, 307, 311, 313, 317, 331, 337, 347, 353, 359, 367, 373, 379, 383,

389, 397, 503, 523, 541, 547, 571, 719, 743, 761, 773, 797, 1103, 1117, 1123, 1129, 1153, 1171, 1303, 1307, 1319, 1361, 1367, 1373, 1723, 1741, 1747, 1753, 1759, 1783, 1789, 1907, 1913, 1931, 1973, 1979, 1997, 2311, 2341, 2347, 2371, 2383, 2389, 2903, 2917, 2953, 2971, 3119, 3137, 3167, 3719, 3761, 3767, 3779, 3797, 4111, 4129, 4153, 4159, 4337, 4373, 4397, 4703, 4723, 4729, 4759, 4783, 4789, 5303, 5323, 5347, 5903, 5923, 5953, 6113, 6131, 6143, 6173, 6197, 6703, 6719, 6737, 6761, 6779, 7103, 7129, 7159, 7307, 7331, 7907, 7919 . The total *primes Cira–Smarandache-Wellin of second order*, from 664579 primes, i.e. all primes up to 10^7 are 12629. The list *Smarandache-Wellin primes in hexadecimal base* generate with commands $L := 1000$ $p := submatrix(prime, 1, L, 1, 1)$ and $Wellin(p, 100, L) =$.

5. Primes that have numbers of 3 digits primes are *Cira–Smarandache–Wellin primes of third order*. The list the *Cira–Smarandache–Wellin primes of third order*, from 10^3 primes, is: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113, 127, 131, 137, 139, 149, 151, 157, 163, 167, 173, 179, 181, 191, 193, 197, 199, 211, 223, 227, 229, 233, 239, 241, 251, 257, 263, 269, 271, 277, 281, 283, 293, 307, 311, 313, 317, 331, 337, 347, 349, 353, 359, 367, 373, 379, 383, 389, 397, 401, 409, 419, 421, 431, 433, 439, 443, 449, 457, 461, 463, 467, 479, 487, 491, 499, 503, 509, 521, 523, 541, 547, 557, 563, 569, 571, 577, 587, 593, 599, 601, 607, 613, 617, 619, 631, 641, 643, 647, 653, 659, 661, 673, 677, 683, 691, 701, 709, 719, 727, 733, 739, 743, 751, 757, 761, 769, 773, 787, 797, 809, 811, 821, 823, 827, 829, 839, 853, 857, 859, 863, 877, 881, 883, 887, 907, 911, 919, 929, 937, 941, 947, 953, 967, 971, 977, 983, 991, 997, 2003, 2011, 2017, 2029, 2053, 2083, 2089, 2113, 2131, 2137, 2179, 2239, 2251, 2269, 2281, 2293, 2311, 2347, 2383, 2389, 2467, 2503, 2521, 2557, 2593, 2617, 2647, 2659, 2677, 2683, 2719, 2797, 2857, 2887, 2953, 2971, 3011, 3019, 3023, 3037, 3041, 3061, 3067, 3079, 3083, 3089, 3109, 3137, 3163, 3167, 3181, 3191, 3229, 3251, 3257, 3271, 3307, 3313, 3331, 3347, 3359, 3373, 3389, 3433, 3449, 3457, 3461, 3463, 3467, 3491, 3499, 3541, 3547, 3557, 3571, 3593, 3607, 3613, 3617, 3631, 3643, 3659, 3673, 3677, 3691, 3701, 3709, 3719, 3727, 3733, 3739, 3761, 3769, 3797, 3821, 3823, 3853, 3863, 3877, 3881, 3907, 3911, 3919, 3929, 3947, 3967, 5003, 5011, 5023, 5059, 5101, 5107, 5113, 5167, 5179, 5197, 5227, 5233, 5281, 5347, 5419, 5431, 5443, 5449, 5479, 5503, 5521, 5557, 5563, 5569, 5641, 5647, 5653, 5659, 5683, 5701, 5743, 5821, 5827, 5839, 5857, 5881, 5953, 7013, 7019, 7043, 7079, 7103, 7109, 7127, 7151, 7193, 7211, 7229, 7283, 7307, 7331, 7349, 7433, 7457, 7487, 7499, 7523, 7541, 7547, 7577, 7607, 7643, 7673, 7691, 7727, 7757, 7823, 7829, 7853, 7877, 7883, 7907, 7919 . The total *Cira–Smarandache–Wellin primes of third order*, from 664579 primes, i.e. all primes up to

10^7 are 22716. The list *Smarandache-Wellin primes in hexadecimal base* generate with commands $L := 1000$ $p := submatrix(prime, 1, L, 1, 1)$ and $Wellin(p, 1000, L) =$.

In the same general conditions of a given sequence, one screens it selecting only its terms whose groups of digits hold the property (or relationship involving the groups of digits) p. A group of digits may contain one or more digits, but not the whole term.

Chapter 5

Sequences Applied in Science

5.1 Unmatter Sequences

Unmatter is formed by combinations of matter and antimatter that bind together, or by long–range mixture of matter and antimatter forming a weakly–coupled phase.

And Unmmatter Plasma is a novel form of plasma, exclusively made of matter and its antimatter counterpart.

5.1.1 Unmatter Combinations

Unmatter combinations as pairs of quarks (q) and antiquarks (a), for $q \geq 1$ and $a \geq 1$. Each combination has $n = q + a \geq 2$ quarks and antiquarks which preserve the colorless, [Smarandache, 2004a,b, 2005], [Sloane, 2014, A181633].

1. if $n = 2$, we have: qa (biquark – for example the mesons and antimessons), so the pair is $(1,1)$;
2. if $n = 3$ we have no unmatter combination, so the pair is $(0,0)$;
3. if $n = 4$, we have $qqaa$ (tetraquark), the pair is $(2,2)$;
4. if $n = 5$, we have $qqqqa$, $qaaaa$ (pentaquark), so the pairs are $(4,1)$ and $(1,4)$;
5. if $n = 6$, we have $qqqaaa$ (hexaquark), whence $(3,3)$;
6. if $n = 7$, we have $qqqqqaa$, $qqaaaaa$ (septiquark), whence $(5,2)$, $(2,5)$;
7. if $n = 8$, we have $qqqqqqqa$, $qqqqaaaa$, $qaaaaaaa$ (octoquark), whence $(7,1)$, $(4,4)$, $(1,7)$;

8. if $n = 9$, we have $qqqqqqaaa$, $qqqaaaaaa$ (nonaquark), whence $(6,3)$, $(3,6)$;

9. if $n = 10$, we have $qqqqqqqqaa$, $qqqqqaaaaa$, $qqaaaaaaaa$ (decaquark), whence $(8,2)$, $(5,5)$, $(2,8)$;

From the conditions

$$\begin{cases} q + a = n \\ q - a = 3k \end{cases} \tag{5.1}$$

result the solutions

$$\begin{cases} a = \dfrac{n-3k}{2} \\ q = \dfrac{n+3k}{2} \end{cases} , \tag{5.2}$$

that must be $a, q \in \mathbb{N}^*$, then result that

$$-\left\lfloor \frac{n-2}{3} \right\rfloor \le k \le \left\lfloor \frac{n-2}{3} \right\rfloor \text{ and } k \in \mathbb{Z} . \tag{5.3}$$

Program 5.1. for generate the unmatter combinations.

$$UC(n,z) := \left| \begin{array}{l} return\ \ "Error."\ \ if\ n < 2 \\ return\ \ (1\ \ 1)^{\mathrm{T}}\ \ if\ n{=}2 \\ return\ \ (0\ \ 0)^{\mathrm{T}}\ \ if\ n{=}3 \wedge z{=}1 \\ i \leftarrow floor\left(\dfrac{n}{3}\right)\ \ if\ z{=}0 \\ i \leftarrow floor\left(\dfrac{n-2}{3}\right)\ \ if\ z{=}1 \\ j \leftarrow 1 \\ for\ k \in -i..i \\ \quad \left| \begin{array}{l} a \leftarrow \dfrac{n-3k}{2} \\ if\ a = trunc(a) \\ \quad \left| \begin{array}{l} qa_j \leftarrow a \\ j \leftarrow j+1 \end{array} \right. \\ q \leftarrow \dfrac{n+3k}{2} \\ if\ q = trunc(q) \\ \quad \left| \begin{array}{l} qa_j \leftarrow q \\ j \leftarrow j+1 \end{array} \right. \end{array} \right. \\ return\ \ qa \end{array} \right.$$

In this program was taken into account formulas 5.2, 5.3 and 5.4.

Program 5.2. for generate the unmatter sequences, for $n = \alpha, \alpha+1, \ldots, \beta$, where $\alpha, \beta \in \mathbb{N}^*$, $\alpha < \beta$.

$$UCS(\alpha,\beta,z) := \left| \begin{array}{l} S \leftarrow UC(\alpha,z) \\ for\ n \in \alpha+1..\beta \\ \quad S \leftarrow stack\big(S, UC(n,z)\big) \\ return\ \ S \end{array} \right.$$

For $\alpha = 2$ and $\beta = 30$, the unmattter sequence is: $UCS(\alpha,\beta,1)^{\mathrm{T}} \rightarrow$ 1, 1, 0, 0, 2, 2, 4, 1, 1, 4, 3, 3, 5, 2, 2, 5, 7, 1, 4, 4, 1, 7, 6, 3, 3, 6, 8, 2, 5, 5, 2, 8, 10, 1, 7, 4, 4, 7, 1, 10, 9, 3, 6, 6, 3, 9, 11, 2, 8, 5, 5, 8, 2, 11, 13, 1, 10, 4, 7, 7, 4, 10, 1, 13, 12, 3, 9, 6, 6, 9, 3, 12, 14, 2, 11, 5, 8, 8, 5, 11, 2, 14, 16, 1, 13, 4, 10, 7, 7, 10, 4, 13, 1, 16, 15, 3, 12, 6, 9, 9, 6, 12, 3, 15, 17, 2, 14, 5, 11, 8, 8, 11, 5, 14, 2, 17, 19, 1, 16, 4, 13, 7, 10, 10, 7, 13, 4, 16, 1, 19 .

5.1.2 Unmatter Combinations of Quarks and Antiquarks

Unmatter combinations of quarks and antiquarks of length $n \geq 1$ that preserve the colorless.

There are 6 types of quarks: Up, Down, Top, Bottom, Strange, Charm and 6 types of antiquarks: *Up*^, *Down*^, *Top*^, *Bottom*^, *Strange*^, *Charm*^.

1. For $n = 1$, we have no unmatter combination;

2. For combinations of 2 we have: *qa* (unmatter biquark), [mesons and antimesons]; the number of all possible unmatter combinations will be $6 \times 6 = 36$, but not all of them will bind together. It is possible to combine an entity with its mirror opposite and still bound them, such as: *uu*^, *dd*^, *ss*^, *cc*^, *bb*^ which form mesons. It is possible to combine, *unmatter* + *unmatter* = *unmatter*, as in *ud*^ + *us*^ = *uudd*^*ss*^ (of course if they bind together)

3. For combinations of 7 we have: *qqqqqaa*, *qqaaaaa* (unmatter septiquarks); the number of all possible unmatter combinations will be $6^5 \times 6^2 + 6^2 \times 6^5 = 559872$, but not all of them will bind together.

4. For combinations of 8 we have: *qqqqaaaa*, *qqqqqqqa*, *qaaaaaaa* (unmatter octoquarks); the number of all possible unmatter combinations will be $6^7 \times 6^1 + 6^4 \times 6^4 + 6^1 \times 6^7 = 5038848$, but not all of them will bind together.

5. For combinations of 9 we have: *qqqqqqaaa*, *qqqaaaaaa* (unmatter non-aquarks); the number of all possible unmatter combinations will be $6^6 \times 6^3 + 6^3 \times 6^6 = 2 \times 6^9 = 20155392$, but not all of them will bind together.

6. For combinations of 10 we have: *qqqqqqqqaa*, *qqqqqaaaaa*, *qqaaaaaaaa* (unmatter decaquarks); the number of all possible unmatter combinations will be $3 \times 6^{10} = 181398528$, but not all of them will bind together.

7. Etc.

Program 5.3. for generate the sequence of unmatter combinations of quarks and antiquarks.

$$UCqa(\alpha,\beta,z) := \left| \begin{array}{l} j \leftarrow 2 \\ for\ n \in \alpha..\beta \\ \quad \left| \begin{array}{l} qa \leftarrow UC(n,z) \\ t_j \leftarrow 0 \\ for\ k \in 1,3..last(qa) \\ \quad t_j \leftarrow t_j + 6^{qa_k + qa_{k+1}} \\ j \leftarrow j+1 \end{array} \right. \\ t_3 \leftarrow 0\ \ if\ z{=}1 \\ return\ \ t \end{array} \right.$$

For $\alpha = 2$ and $\beta = 30$, the sequence of unmatter combinations of quarks and antiquarks is:

$UCqa(\alpha,\beta,1)^{\mathrm{T}} \rightarrow$ 0, 36, 0, 1296, 15552, 46656, 559872, 5038848, 20155392, 181398528, 1451188224, 6530347008, 52242776064, 391820820480, 1880739938304, 14105549537280, 101559956668416, 507799783342080, 3656158440062976, 25593109080440832, 131621703842267136, 921351926895869952, 6317841784428822528, 33168669368251318272, 227442304239437611008, 1535235553616203874304, 8187922952619753996288, 55268479930183339474944, 368456532867888929832960, 1989665277486600221097984 .

I wonder if it is possible to make infinitely many combinations of quarks / antiquarks and leptons / antileptons … . Unmatter can combine with matter and / or antimatter and the result may be any of these three. Some unmatter could be in the strong force, hence part of hadrons.

5.1.3 Colorless Combinations as Pairs of Quarks and Antiquarks

Colorless combinations as pairs of quarks and antiquarks, for $q, a \geq 0$;

1. if $n = 2$, we have: *qa* (biquark – for example the mesons and antimessons), whence the pair $(1,1)$;
2. if $n = 3$, we have: *qqq*, *aaa* (triquark – for example the baryons and antibaryons), whence the pairs $(3,0)$, $(0,3)$;
3. if $n = 4$, we have *qqaa* (tetraquark), whence the pair $(2,2)$;
4. if $n = 5$, we have *qqqqa*, *qaaaa* (pentaquark), whence the pairs $(4,1)$, $(1,4)$;
5. if $n = 6$, we have *qqqqqq*, *qqqaaa*, *aaaaaa* (hexaquark), whence the pairs $(6,0)$, $(3,3)$, $(0,6)$;
6. if $n = 7$, we have *qqqqqaa*, *qqaaaaa* (septiquark), whence the pairs $(5,2)$, $(2,5)$;
7. if $n = 8$, we have *qqqqqqqa*, *qqqqaaaa*, *qaaaaaaa* (octoquark), whence the pairs $(7,1)$, $(4,4)$, $(1,7)$;
8. if $n = 9$, we have *qqqqqqqqq*, *qqqqqqaaa*, *qqqaaaaaa*, *aaaaaaaaa* (nonaquark), whence the pairs $(9,0)$, $(6,3)$, $(3,6)$, $(0,9)$;
9. if $n = 10$, we have *qqqqqqqqaa*, *qqqqqaaaaa*, *qqaaaaaaaa* (decaquark), whence the pairs $(8,2)$, $(5,5)$, $(2,8)$; There are symmetric pairs.

From the conditions 5.1 result the solutions 5.2, that must be $a, q \in \mathbb{N}$, then result that
$$-\left\lfloor\frac{n}{3}\right\rfloor \leq k \leq \left\lfloor\frac{n}{3}\right\rfloor \text{ and } k \in \mathbb{Z}\,. \tag{5.4}$$

For $\alpha = 2$ and $\beta = 30$, the unmattter sequence is: $UCS(\alpha,\beta,0)^{\mathrm{T}} \to$ 1, 1, 3, 0, 0, 3, 2, 2, 4, 1, 1, 4, 6, 0, 3, 3, 0, 6, 5, 2, 2, 5, 7, 1, 4, 4, 1, 7, 9, 0, 6, 3, 3, 6, 0, 9, 8, 2, 5, 5, 2, 8, 10, 1, 7, 4, 4, 7, 1, 10, 12, 0, 9, 3, 6, 6, 3, 9, 0, 12, 11, 2, 8, 5, 5, 8, 2, 11, 13, 1, 10, 4, 7, 7, 4, 10, 1, 13, 15, 0, 12, 3, 9, 6, 6, 9, 3, 12, 0, 15, 14, 2, 11, 5, 8, 8, 5, 11, 2, 14, 16, 1, 13, 4, 10, 7, 7, 10, 4, 13, 1, 16, 18, 0, 15, 3, 12, 6, 9, 9, 6, 12, 3, 15, 0, 18, 17, 2, 14, 5, 11, 8, 8, 11, 5, 14, 2, 17, 19, 1, 16, 4, 13, 7, 10, 10, 7, 13, 4, 16, 1, 19 , where *UCS* is the program 5.2.

In order to save the colorless combinations prevailed in the Theory of Quantum Chromodynamics (QCD) of quarks and antiquarks in their combinations when binding, we devised the following formula, [Smarandache, 2004a, 2005]:

q is congruent with a, modulo 3; where q = number of quarks and a = number of antiquarks. To justify this formula we mention that 3 quarks form a colorless combination and any multiple of three combination of quarks too, i.e. 6, 9, 12, etc. quarks. In a similar way, 3 antiquarks form a colorless combination and any multiple of three combination of antiquarks too, i.e. 6, 9, 12, etc. antiquarks.

- If n is even, $n = 2k$, then its pairs are: $(k+3m, k-3m)$, where m is an integer such that both $k+3m \geq 0$ and $k-3m \geq 0$.
- If n is odd, $n = 2k+1$, then its pairs are: $(k+3m+2, k-3m-1)$, where m is an integer such that both $k+3m+2 \geq 0$ and $k-3m-1 \geq 0$.

5.1.4 Colorless Combinations of Quarks and Qntiquarks of Length $n \geq 1$

Colorless combinations of quarks and antiquarks of length $n \geq 1$, for $q \geq 0$ and $a \geq 0$.

Comment:

- If $n = 1$ there is no colorless combination.
- If $n = 2$ we have qa (quark antiquark), so a pair $(1,1)$; since a quark can be *Up*, *Down*, *Top*, *Bottom*, *Strange*, *Charm* while an antiquark can be *Up*^, *Down*^, *Top*^, *Bottom*^, *Strange*^, *Charm*^ then we have $6 \times 6 = 36$ combinations.
- If $n = 3$ we have *qqq* and *aaa*, thus two pairs $(3,0)$, $(0,3)$, i.e. $2 \times 6^3 = 432$.
- If $n = 4$, we have *qqaa*, so the pair $(2,2)$, i.e. $6^4 = 1296$.

For $\alpha = 2$ and $\beta = 30$, the sequence of unmatter combinations of quarks and antiquarks is:

$$
\begin{aligned}
UCqa(\alpha,\beta,0)^{\mathrm{T}} \to\ & 0,\ 36,\ 432,\ 1296,\ 15552,\ 139968,\ 559872,\\
& 5038848,\ 40310784,\ 181398528,\ 1451188224,\ 10883911680,\\
& 52242776064,\ 391820820480,\ 2821109907456,\ 14105549537280,\\
& 101559956668416,\ 710919696678912,\ 3656158440062976,\\
& 25593109080440832,\ 175495605123022848,\ 921351926895869952,\\
& 6317841784428822528,\ 42645432044894552064,\\
& 227442304239437611008,\ 1535235553616203874304,\\
& 10234903690774692495360,\ 55268479930183339474944,\\
& 368456532867888929832960,\ 2431813116928066936897536\ ,
\end{aligned}
$$

where *UCqa* is the program 5.3.

5.2 Convex Polyhedrons

A convex polyhedron can be defined algebraically as the set of solutions to a system of linear inequalities

$$M \cdot x \leq b$$

where M is a real $m \times 3$ matrix and b is a real m–vector. Although usage varies, most authors additionally require that a solution be bounded for it to qualify as a convex polyhedron. A convex polyhedron may be obtained from an arbitrary set of points by computing the convex hull of the points.

Explicit examples are given in the following table:

1. Tetrahedron, $m = 4$ and

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix} \cdot x \leq \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix};$$

2. Cube, $m = 6$ and

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix} \cdot x \leq \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix};$$

3. Octahedron, $m = 8$ and

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \\ -1 & -1 & -1 \end{pmatrix} \cdot x \leq \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

Geometrically, a convex polyhedron can be defined as a polyhedron for which a line connecting any two (noncoplanar) points on the surface always lies in the interior of the polyhedron. Every convex polyhedron can be represented in the plane or on the surface of a sphere by a 3–connected planar graph

(called a polyhedral graph). Conversely, by a theorem of Steinitz as restated by Grünbaum, every 3-connected planar graph can be realized as a convex polyhedron (Duijvestijn and Federico 1981).

1. Given n points in space, four by four non-coplanar, find the maximum number $M(n)$ of points which constitute the vertexes of a convex polyhedron, [Tomescu, 1983]. Of course, $M(n) \geq 4$.

2. Given n points in space, four by four non-coplanar, find the minimum number $N(n) \geq 5$ such that: any $N(n)$ points among these do not constitute the vertexes of a convex polyhedron. Of course, $N(n)$ may not exist.

Chapter 6

Constants

6.1 Smarandache Constants

In Mathworld website, [Weisstein, 2015g], one finds the following constants related to the Smarandache function.

Observation 6.1. All definitions use S for denoting Smarandache function 2.67.

The *Smarandache constant* is the smallest solution to the generalized Andrica's conjecture, $x \approx 0.567148\ldots$, [Sloane, 2014, A038458].

Equation solutions

$$p^x - (p+g)^x = 1\ , \quad p \in \mathbb{P}_{\geq 2}\ , \tag{6.1}$$

where $g = g_n = p_{n+1} - p_n$ is the gap between two consecutive prime numbers.

The solutions to equation (6.1) in ascending order using the maximal gaps, [Oliveira e Silva, 2014], [Cira, 2014].

Table 6.1: Equation (6.1) solutions

p	g	solution for equation (6.1)
113	14	0.5671481305206224...
1327	34	0.5849080865740931...
7	4	0.5996694211239202...
23	6	0.6042842019286720...
523	18	0.6165497314215637...
1129	22	0.6271418980644412...
887	20	0.6278476315319166...
31397	72	0.6314206007048127...

Continued on next page

p	g	solution for equation (6.1)
89	8	0.6397424613256825...
19609	52	0.6446915279533268...
15683	44	0.6525193297681189...
9551	36	0.6551846556887808...
155921	86	0.6619804741301879...
370261	112	0.6639444999972240...
492113	114	0.6692774164975257...
360653	96	0.6741127001176469...
1357201	132	0.6813839139412406...
2010733	148	0.6820613370357171...
1349533	118	0.6884662952427394...
4652353	154	0.6955672852207547...
20831323	210	0.7035651178160084...
17051707	180	0.7088121412466053...
47326693	220	0.7138744163020114...
122164747	222	0.7269826061830018...
3	2	0.7271597432435757...
191912783	248	0.7275969819805509...
189695659	234	0.7302859105830866...
436273009	282	0.7320752818323865...
387096133	250	0.7362578381533295...
1294268491	288	0.7441766589716590...
1453168141	292	0.7448821415605216...
2300942549	320	0.7460035467176455...
4302407359	354	0.7484690049408947...
3842610773	336	0.7494840618593505...
10726904659	382	0.7547601234459729...
25056082087	456	0.7559861641728429...
42652618343	464	0.7603441937898209...
22367084959	394	0.7606955951728551...
20678048297	384	0.7609716068556747...
127976334671	468	0.7698203623795380...
182226896239	474	0.7723403816143177...
304599508537	514	0.7736363009251175...
241160624143	486	0.7737508697071668...
303371455241	500	0.7745991865337681...
297501075799	490	0.7751693424982924...

Continued on next page

p	g	solution for equation (6.1)
461690510011	532	0.7757580339651479...
416608695821	516	0.7760253389165942...
614487453523	534	0.7778809828805762...
1408695493609	588	0.7808871027951452...
1346294310749	582	0.7808983645683428...
2614941710599	652	0.7819658004744228...
1968188556461	602	0.7825687226257725...
7177162611713	674	0.7880214782837229...
13829048559701	716	0.7905146362137986...
19581334192423	766	0.7906829063252424...
42842283925351	778	0.7952277512573828...
90874329411493	804	0.7988558653770882...
218209405436543	906	0.8005126614171458...
171231342420521	806	0.8025304565279002...
1693182318746371	1132	0.8056470803187964...
1189459969825483	916	0.8096231085041140...
1686994940955803	924	0.8112057874892308...
43841547845541060	1184	0.8205327998695296...
55350776431903240	1198	0.8212591131062218...
80873624627234850	1220	0.8224041089823987...
218034721194214270	1248	0.8258811322716928...
352521223451364350	1328	0.8264955008480679...
1425172824437699300	1476	0.8267652954810718...
305405826521087900	1272	0.8270541728027422...
203986478517456000	1224	0.8271121951019150...
418032645936712100	1370	0.8272229385637846...
401429925999153700	1356	0.8272389079572986...
804212830686677600	1442	0.8288714147741382...
2	1	1

1. The first Smarandache constant is defined as

$$S_1 = \sum_{n=2}^{\infty} \frac{1}{S(n)!} = 1.09317\ldots\,, \tag{6.2}$$

[Sloane, 2014, A048799]. Cojocaru and Cojocaru [1996a] prove that S_1 exists and is bounded by $0.717 < S_1 < 1.253$.

2. Cojocaru and Cojocaru [1996b] prove that the second Smarandache constant

$$S_2 = \sum_{n=2}^{\infty} \frac{S(n)}{n!} \approx 1.71400629359162\ldots\,, \tag{6.3}$$

[Sloane, 2014, A048834] is an irrational number.

3. Cojocaru and Cojocaru [1996c] prove that the series

$$S_3 = \sum_{n=2}^{\infty} \frac{1}{\prod_{m=2}^{n} S(m)} \approx 0.719960700043708 \tag{6.4}$$

converges to a number $0.71 < S_3 < 1.01$.

4. Series

$$S_4(\alpha) = \sum_{n=2}^{\infty} \frac{n^{\alpha}}{\prod_{m=2}^{n} S(m)}\,. \tag{6.5}$$

converges for a fixed real number $a \geq 1$. The values for small a are

$$S_4(1) \approx 1.72875760530223\ldots\,; \tag{6.6}$$
$$S_4(2) \approx 4.50251200619297\ldots\,; \tag{6.7}$$
$$S_4(3) \approx 13.0111441949445\ldots\,, \tag{6.8}$$
$$S_4(4) \approx 42.4818449849626\ldots\,; \tag{6.9}$$
$$S_4(5) \approx 158.105463729329\ldots\,, \tag{6.10}$$

[Sloane, 2014, A048836, A048837, A048838].

5. Sandor [1997] shows that the series

$$S_5 = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} S(n)}{n!} \tag{6.11}$$

converges to an irrational.

6. Burton [1995] and Dumitrescu and Seleacu [1996] show that the series

$$S_6 = \sum_{n=2}^{\infty} \frac{S(n)}{(n+1)!} \tag{6.12}$$

converges.

7. Dumitrescu and Seleacu [1996] show that the series
$$S_7 = \sum_{n=r}^{\infty} \frac{S(n)}{(n+r)!}\,, \tag{6.13}$$
for $r \in \mathbb{N}$, and
$$S_8 = \sum_{n=r}^{\infty} \frac{S(n)}{(n-r)!} \tag{6.14}$$
for $r \in \mathbb{N}^*$, converges.

8. Dumitrescu and Seleacu [1996] show that
$$S_9 = \sum_{n=2}^{\infty} \frac{1}{\displaystyle\sum_{m=2}^{n} \frac{S(m)}{m!}} \tag{6.15}$$
converges.

9. Burton [1995], Dumitrescu and Seleacu [1996] show that the series
$$S_{10} = \sum_{n=2}^{\infty} \frac{1}{\big(S(n)\big)^{\alpha} \cdot \sqrt{S(n)!}} \tag{6.16}$$
and
$$S_{11} = \sum_{n=2}^{\infty} \frac{1}{\big(S(n)\big)^{\alpha} \cdot \sqrt{(S(n)+1)!}} \tag{6.17}$$
converge for $\alpha \in \mathbb{N}$, $\alpha > 1$.

6.2 Erdös–Smarandache Constants

The authors did not prove the convergence towards each constant. We let it as possible research for the interested readers. With the program *ES*, 3.49, calculate vector top 100 terms numbers containing Erdös–Smarandache, $es = ES(2,130)$. The vector es has 100 terms for $n := last(es) = 100$ and last term vector es has the value 130, because $es_{\text{last}(es)} = 130$.

1. The first constant Erdös–Smarandache is defined as
$$ES_1 = \sum_{k=1}^{\infty} \frac{1}{es_k!} \approx \sum_{k=1}^{n} \frac{1}{es_k!} = 0.6765876023854308\ldots\,, \tag{6.18}$$
it is well approximated because
$$\frac{1}{es_n!} = 1.546 \cdot 10^{-220}\,.$$

2. The second constant Erdös–Smarandache is defined as
$$ES_2 = \sum_{k=1}^{\infty} \frac{es_k}{k!} \approx \sum_{k=1}^{n} \frac{es_k}{k!} = 4.658103698740189\ldots\ , \tag{6.19}$$
it is well approximated because
$$\frac{es_n}{n!} = 1.393 \cdot 10^{-156}\ .$$

3. The third constant Erdös–Smarandache is defined as
$$ES_3 = \sum_{k=1}^{\infty} \frac{1}{\prod_{j=1}^{k} es_j} \approx \sum_{k=1}^{n} \frac{1}{\prod_{j=1}^{k} es_j} = 0.7064363838861719\ldots\ , \tag{6.20}$$
it is well approximated because
$$\frac{1}{\prod_{j=1}^{n} es_j} = 3.254 \cdot 10^{-173}\ .$$

4. Series
$$ES_4(\alpha) = \sum_{k=1}^{\infty} \frac{k^\alpha}{\prod_{j=1}^{k} es_j}\ , \tag{6.21}$$
then
   - The case $\alpha = 1$
   $$ES_4(1) \approx \sum_{k=1}^{n} \frac{k}{\prod_{j=1}^{k} es_j} = 0.9600553300834916\ldots$$
   it is well approximated because $\dfrac{n}{\prod_{j=1}^{n} es_j} = 3.254 \cdot 10^{-171}$,
   - The case $\alpha = 2$
   $$ES_4(2) \approx \sum_{k=1}^{n} \frac{k^2}{\prod_{j=1}^{k} es_j} = 1.5786465190659933\ldots$$
   it is well approximated because $\dfrac{n^2}{\prod_{j=1}^{n} es_j} = 3.254 \cdot 10^{-169}$,

- The case $\alpha = 3$

$$ES_4(3) \approx \sum_{k=1}^{n} \frac{k^3}{\prod_{j=1}^{k} es_j} = 3.208028767543241\ldots$$

it is well approximated because $\dfrac{n^3}{\prod_{j=1}^{n} es_j} = 3.254 \cdot 10^{-167}$,

- The case $\alpha = 4$

$$ES_4(4) \approx \sum_{k=1}^{n} \frac{k^4}{\prod_{j=1}^{k} es_j} = 7.907663276289289\ldots$$

it is well approximated because $\dfrac{n^4}{\prod_{j=1}^{n} es_j} = 3.254 \cdot 10^{-165}$,

- The case $\alpha = 5$

$$ES_4(5) \approx \sum_{k=1}^{n} \frac{k^5}{\prod_{j=1}^{k} es_j} = 22.86160508982205\ldots$$

it is well approximated because $\dfrac{n^5}{\prod_{j=1}^{n} es_j} = 3.254 \cdot 10^{-163}$.

5. Series

$$ES_5 = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} es_k}{k!} \approx \sum_{k=1}^{n} \frac{(-1)^{k+1} es_k}{k!} = 1.1296727326811478 \tag{6.22}$$

it is well approximated because

$$\frac{(-1)^{n+1} es_n}{n!} = -1.393 \cdot 10^{-156} .$$

6. Series

$$ES_6 = \sum_{k=1}^{\infty} \frac{es_k}{(k+1)!} \approx \sum_{k=1}^{n} \frac{es_k}{(k+1)!} = 1.7703525971096077 , \tag{6.23}$$

it is well approximated because

$$\frac{es_n}{(n+1)!} = 1.379 \cdot 10^{-158} \,.$$

7. Series

$$ES_7(r) = \sum_{k=r}^{\infty} \frac{es_k}{(k+r)!} \,, \tag{6.24}$$

with $r \in \mathbb{N}^*$, then

- The case $r = 1$ $\big($be noticed as $ES_7(1) = ES_6\big)$

$$ES_7(1) \approx \sum_{k=1}^{n} \frac{es_k}{(k+1)!} = 1.7703525971096077 \,,$$

it is well approximated because $\dfrac{es_n}{(n+1)!} = 1.379 \cdot 10^{-158}$,

- The case $r = 2$

$$ES_7(2) \approx \sum_{k=2}^{n} \frac{es_k}{(k+2)!} = 0.17667118527354841 \,,$$

it is well approximated because $\dfrac{es_n}{(n+2)!} = 1.352 \cdot 10^{-160}$,

- The case $r = 3$

$$ES_7(3) \approx \sum_{k=3}^{n} \frac{es_k}{(k+3)!} = 0.0083394778946466 \,,$$

it is well approximated because $\dfrac{es_n}{(n+3)!} = 1.313 \cdot 10^{-162}$,

8. Series

$$ES_8(r) = \sum_{k=r}^{\infty} \frac{es_k}{(k-r)!} \,, \tag{6.25}$$

with $r \in \mathbb{N}^*$, then

- The case $r = 1$

$$ES_8(1) \approx \sum_{k=1}^{n} \frac{es_k}{(k-1)!} = 8.893250907189714 \,,$$

it is well approximated because $\dfrac{es_n}{(n-1)!} = 1.393 \cdot 10^{-154}$,

- The case $r = 2$

$$ES_8(2) \approx \sum_{k=2}^{n} \frac{es_k}{(k-2)!} = 12.69625798917767\ ,$$

it is well approximated because $\dfrac{es_n}{(n-2)!} = 1.379 \cdot 10^{-152}$,

- The case $r = 3$

$$ES_8(3) \approx \sum_{k=3}^{n} \frac{es_k}{(k-3)!} = 16.756234041646312\ ,$$

it is well approximated because $\dfrac{es_n}{(n-3)!} = 1.351 \cdot 10^{-150}$.

9. Series

$$ES_9 = \sum_{k=1}^{\infty} \frac{1}{\sum_{j=1}^{k} es_j!} \approx \sum_{k=1}^{n} \frac{1}{\sum_{j=1}^{k} es_j!} = 0.6341618804985396\ , \tag{6.26}$$

it is well approximated because

$$\frac{1}{\sum_{j=1}^{n} es_j!} = 1.535 \cdot 10^{-220}\ .$$

10. Series

$$ES_{10}(\alpha) = \sum_{k=1}^{\infty} \frac{1}{es_k^{\alpha} \sqrt{es_k!}}\ , \tag{6.27}$$

then

- The case $\alpha = 1$

$$ES_{10}(1) \approx \sum_{k=1}^{n} \frac{1}{es_k \sqrt{es_k!}} = 0.5161853069935946\ ,$$

it is well approximated because $\dfrac{1}{es_k \sqrt{es_k!}} = 9.566 \cdot 10^{-113}$,

- The case $\alpha = 2$

$$ES_{10}(2) \approx \sum_{k=1}^{n} \frac{1}{es_k^2 \sqrt{es_k!}} = 0.22711843820442665\ ,$$

it is well approximated because $\dfrac{1}{es_k^2 \sqrt{es_k!}} = 7.358 \cdot 10^{-115}$,

- The case $\alpha=3$

$$ES_{10}(3)\approx\sum_{k=1}^{n}\frac{1}{es_k^3\sqrt{es_k!}}=0.10445320547192125\ ,$$

it is well approximated because $\frac{1}{es_k^3\sqrt{es_k!}}=5.66\cdot10^{-117}$.

11. Series

$$ES_{11}(\alpha)=\sum_{k=1}^{\infty}\frac{1}{es_k^{\alpha}\sqrt{(es_k+1)!}}\ , \tag{6.28}$$

then

- The case $\alpha=1$

$$ES_{11}(1)\approx\sum_{k=1}^{n}\frac{1}{es_k\sqrt{(es_k+1)!}}=0.28269850314464495\ ,$$

it is well approximated because $\frac{1}{es_k\sqrt{(es_k+1)!}}=8.357\cdot10^{-114}$,

- The case $\alpha=2$

$$ES_{11}(2)\approx\sum_{k=1}^{n}\frac{1}{es_k^2\sqrt{(es_k+1)!}}=0.1267281413034069\ ,$$

it is well approximated because $\frac{1}{es_k^2\sqrt{(es_k+1)!}}=6.429\cdot10^{-116}$,

- The case $\alpha=3$

$$ES_{11}(3)\approx\sum_{k=1}^{n}\frac{1}{es_k^3\sqrt{(es_k+1)!}}=0.05896925858439456\ ,$$

it is well approximated because $\frac{1}{es_k^3\sqrt{(es_k+1)!}}=4.945\cdot10^{-118}$.

6.3 Smarandache–Kurepa Constants

The authors did not prove the convergence towards each constant. We let it as possible research for the interested readers. With the program *SK*,

2.74, calculate vectors top 25 terms numbers containing Smarandache–Kurepa, *sk1* = $SK(1,p)$, *sk2* = $SK(2,p)$ and *sk3* = $SK(3,p)$, where

$$p = (2\ \ 3\ \ 5\ \ 7\ \ 11\ \ 13\ \ 17\ \ 19\ \ 23\ \ 29\ \ 31\ \ 37\ \ 41\ \ 43\ \ 47\ \ 53\ \ 59\ \ 61\ \ 67\ \ 71\ \ 73\ \ 79\ \ 83\ \ 89\ \ 97)^{\mathrm{T}}\,.$$

Vectors *sk1* (2.93), *sk2* (2.95) and *sk3* (2.97) has 25 terms.

1. The first constant Smarandache–Kurepa is defined as

$$SK_1 = \sum_{k=1}^{\infty} \frac{1}{sk_k!}\,. \tag{6.29}$$

Program 6.2. for the approximation of SK_1.

$$SK_1(sk) := \left| \begin{array}{l} SK \leftarrow 0 \\ for\ k \in 1..\mathrm{last}(sk) \\ \quad SK \leftarrow SK + \dfrac{1}{sk_k!}\ \ if\ sk_k \neq -1 \\ return\ SK \end{array} \right.$$

Thus is obtained:

- $SK_1(sk1)$*float*$,20 \rightarrow 0.55317460526232666816\ldots$, it is well approximated because
$$\frac{1}{sk1_{last(sk1)}!} = 1.957 \cdot 10^{-20}\,;$$
- $SK_1(sk2)$*float*$,20 \rightarrow 0.55855654987879293658\ldots$, it is well approximated because
$$\frac{1}{sk2_{last(sk2)}!} = 3.8 \cdot 10^{-36}\,;$$
- $SK_1(sk3)$*float*$,20 \rightarrow 0.55161215327881994551\ldots$, it is well approximated because
$$\frac{1}{sk3_{last(sk3)}!} = 7.117 \cdot 10^{-52}\,.$$

2. The second constant Smarandache–Kurepa is defined as

$$SK_2 = \sum_{k=1}^{\infty} \frac{sk_k}{k!}\,. \tag{6.30}$$

Program 6.3. for the approximation of SK_2.

$$SK_2(sk) := \left| \begin{array}{l} SK \leftarrow 0 \\ for\ k \in 1..\mathrm{last}(sk) \\ \quad SK \leftarrow SK + \dfrac{sk_k}{k!}\ \ if\ sk_k \neq -1 \\ return\ \ SK \end{array} \right.$$

Thus is obtained:

- $SK_2(sk1)\mathit{float}, 20 \rightarrow 2.967851980516919686\ldots$, it is well approximated because
$$\frac{sk1_{last(sk1)}}{last(sk1)!} = 1.354 \cdot 10^{-24}\ ;$$
- $SK_2(sk2)\mathit{float}, 20 \rightarrow 5.5125891876109912425\ldots$, it is well approximated because
$$\frac{sk2_{last(sk2)}}{last(sk2)!} = 2.063 \cdot 10^{-24}\ ;$$
- $SK_2(sk3)\mathit{float}, 20 \rightarrow 5.222881245790957486\ldots$, it is well approximated because
$$\frac{sk3_{last(sk3)}}{last(sk3)!} = 2.708 \cdot 10^{-24}\ .$$

3. The third constant Smarandache–Kurepa is defined as
$$SK_3 = \sum_{k=1}^{\infty} \frac{1}{\prod_{j=1}^{k} sk_j}\ . \tag{6.31}$$

Program 6.4. for the approximation of SK_3.

$$SK_3(sk) := \left| \begin{array}{l} SK \leftarrow 0 \\ for\ k \in 1..last(sk) \\ \quad if\ sk_k \neq -1 \\ \quad \left| \begin{array}{l} prod \leftarrow 1 \\ for\ j \in 1..k \\ \quad prod \leftarrow prod \cdot sk_j\ \ if\ sk_j \neq -1 \\ SK \leftarrow SK + \dfrac{1}{prod} \end{array} \right. \\ return\ \ SK \end{array} \right.$$

Thus is obtained:

- $SK_3(sk1)\mathit{float}, 20 \rightarrow 0.65011461681321770674\ldots$,
  it is well approximated because
$$\frac{1}{\left|\prod_{j=1}^{last(sk1)} sk1_j\right|} = 4.301 \cdot 10^{-26}\ ;$$
- $SK_3(sk2)\mathit{float}, 20 \rightarrow 0.62576709781381269162\ldots$,
  it is well approximated because
$$\frac{1}{\left|\prod_{j=1}^{last(sk2)} sk2_j\right|} = 1.399 \cdot 10^{-29}\ ;$$
- $SK_3(sk3)\mathit{float}, 20 \rightarrow 0.60895812831886298 47\ldots$,
  it is well approximated because
$$\frac{1}{\left|\prod_{j=1}^{last(sk3)} sk3_j\right|} = 4.621 \cdot 10^{-31}\ .$$

4. Series
$$SK_4(\alpha) = \sum_{k=1}^{\infty} \frac{k^{\alpha}}{\prod_{j=1}^{k} sk_j}\ . \tag{6.32}$$

Program 6.5. for the approximation of $SK_4(\alpha)$.

$$
SK_4(sk,\alpha) := \left|
\begin{array}{l}
SK \leftarrow 0 \\
for\ k \in 1..last(sk) \\
\quad if\ sk_k \neq -1 \\
\quad \left|
\begin{array}{l}
prod \leftarrow 1 \\
for\ j \in 1..k \\
\quad prod \leftarrow prod \cdot sk_j\ \ if\ sk_j \neq -1 \\
SK \leftarrow SK + \dfrac{k^{\alpha}}{prod}
\end{array}
\right. \\
return\ \ SK
\end{array}
\right.
$$

We define a function that is value the last term of the series (6.32)

$$U4(sk,\alpha) := \frac{last(sk)^{\alpha}}{\left|\prod_{j=1}^{last(sk)} sk_j\right|}\ .$$

Thus is obtained:

- Case $\alpha = 1$,
  - $SK_4(sk1,\alpha)\mathit{float},20 \to 0.98149043761308041099\ldots$,
    it is well approximated because $U4(sk1,\alpha) = 1.075\cdot 10^{-24}$;
  - $SK_4(sk2,\alpha)\mathit{float},20 \to 0.78465913770543477708\ldots$,
    it is well approximated because $U4(sk2,\alpha) = 3.496\cdot 10^{-28}$;
  - $SK_4(sk3,\alpha)\mathit{float},20 \to 0.77461420238539514113\ldots$,
    it is well approximated because $U4(sk3,\alpha) = 1.155\cdot 10^{-29}$;
- Case $\alpha = 2$,
  - $SK_4(sk1,\alpha)\mathit{float},20 \to 2.08681505420554993\ldots$,
    it is well approximated because $U4(sk1,\alpha) = 2.688\cdot 10^{-23}$;
  - $SK_4(sk2,\alpha)\mathit{float},20 \to 1.1883623850019734284\ldots$,
    it is well approximated because $U4(sk2,\alpha) = 8.741\cdot 10^{-27}$;
  - $SK_4(sk3,\alpha)\mathit{float},20 \to 1.2937484108637316754\ldots$,
    it is well approximated because $U4(sk3,\alpha) = 2.888\cdot 10^{-28}$;
- Case $\alpha = 3$,
  - $SK_4(sk1,\alpha)\mathit{float},20 \to 5.9433532880383150933\ldots$,
    it is well approximated because $U4(sk1,\alpha) = 6.721\cdot 10^{-22}$;
  - $SK_4(sk2,\alpha)\mathit{float},20 \to 2.3331345867616929091\ldots$,
    it is well approximated because $U4(sk2,\alpha) = 2.185\cdot 10^{-25}$;
  - $SK_4(sk3,\alpha)\mathit{float},20 \to 3.1599744540262403647\ldots$,
    it is well approximated because $U4(sk3,\alpha) = 7.22\cdot 10^{-27}$;
- Case $\alpha = 4$,
  - $SK_4(sk1,\alpha)\mathit{float},20 \to 20.31367425449123713\ldots$,
    it is well approximated because $U4(sk3,\alpha) = 1.68\cdot 10^{-20}$;
  - $SK_4(sk2,\alpha)\mathit{float},20 \to 6.0605166330133984862\ldots$,
    it is well approximated because $U4(sk3,\alpha) = 5.463\cdot 10^{-24}$;
  - $SK_4(sk3,\alpha)\mathit{float},20 \to 10.61756990155963527\ldots$,
    it is well approximated because $U4(sk3,\alpha) = 1.805\cdot 10^{-25}$.

5. Series
$$SK_5 = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} sk_k}{k!} \ . \tag{6.33}$$

Program 6.6. for the approximation of SK_5.

$$SK_5(sk) := \left| \begin{array}{l} SK \leftarrow 0 \\ for\ k \in 1..last(sk) \\ \quad SK \leftarrow SK + \dfrac{(-1)^{k+1} sk_k}{k!} \ \ if\ sk_k \neq -1 \\ return\ \ SK \end{array} \right.$$

We define a function that is value the last term of the series (6.33)

$$U5(sk) := \frac{(-1)^{last(sk)+1} \cdot sk_{last(sk)}}{last(sk)!} \ .$$

Thus is obtained:

- $SK_5(sk1) float, 20 \to 2.4675046664369494917\ldots$,
  it is well approximated because $U5(sk1) = 1.354 \cdot 10^{-24}$;
- $SK_5(sk2) float, 20 \to 0.15980474179291864895\ldots$,
  it is well approximated because $U5(sk2) = 2.063 \cdot 10^{-24}$;
- $SK_5(sk3) float, 20 \to -1.1304809775475895 44\ldots$,
  it is well approximated because $U5(sk3) = 2.708 \cdot 10^{-24}$.

6. Series
$$SK_6 = \sum_{k=1}^{\infty} \frac{sk_k}{(k+1)!} \ . \tag{6.34}$$

Program 6.7. for the approximation of SK_6.

$$SK_6(sk) := \left| \begin{array}{l} SK \leftarrow 0 \\ for\ k \in 1..last(sk) \\ \quad SK \leftarrow SK + \dfrac{sk_k}{(k+1)!} \ \ if\ sk_k \neq -1 \\ return\ \ SK \end{array} \right.$$

We define a function that is value the last term of the series (6.34)

$$U6(sk) := \frac{sk_{last(sk)}}{(last(sk)+1)!} \ .$$

Thus is obtained:

- $SK_6(sk1)$*float*, $20 \to 1.225145255840940818\ldots$,
  it is well approximated because $U6(sk1) = 5.207 \cdot 10^{-26}$;
- $SK_6(sk2)$*float*, $20 \to 2.0767449920846168598\ldots$,
  it is well approximated because $U6(sk2) = 7.935 \cdot 10^{-26}$;
- $SK_6(sk3)$*float*, $20 \to 2.0422612109916229114\ldots$,
  it is well approximated because $U6(sk3) = 1.041 \cdot 10^{-25}$.

7. Series

$$SK_7(r) = \sum_{k=r}^{\infty} \frac{sk_k}{(k+r)!}\ . \tag{6.35}$$

Program 6.8. for the approximation of $SK_7(r)$.

$$SK_7(sk,r) := \left| \begin{array}{l} SK \leftarrow 0 \\ for\ \ k \in r..last(sk) \\ \quad SK \leftarrow SK + \dfrac{sk_k}{(k+r)!}\ \ if\ sk_k \neq -1 \\ return\ \ SK \end{array} \right.$$

We define a function that is value the last term of the series (6.35)

$$U7(sk,r) := \frac{sk_{last(sk)}}{(last(sk)+r)!}\ .$$

Thus is obtained:

- Case $r = 1$
  - $SK_7(sk1,r)$*float*, $20 \to 1.225145255840940818\ldots$,
    it is well approximated because $U7(sk1,r) = 5.207 \cdot 10^{-26}$;
  - $SK_7(sk2,r)$*float*, $20 \to 2.0767449920846168598\ldots$,
    it is well approximated because $U7(sk2,r) = 7.935 \cdot 10^{-26}$;
  - $SK_7(sk3,r)$*float*, $20 \to 2.0422612109916229114\ldots$,
    it is well approximated because $U7(sk3,r) = 1.041 \cdot 10^{-25}$;
- Case $r = 2$
  - $SK_7(sk1,r)$*float*, $20 \to 0.042873028085536375389\ldots$,
    it is well approximated because $U7(sk1,r) = 1.929 \cdot 10^{-27}$;
  - $SK_7(sk2,r)$*float*, $20 \to 0.25576855146026900397\ldots$,
    it is well approximated because $U7(sk2,r) = 2.939 \cdot 10^{-27}$;
  - $SK_7(sk3,r)$*float*, $20 \to 0.25678656434224640866\ldots$,
    it is well approximated because $U7(sk3,r) = 3.857 \cdot 10^{-27}$;

- Case $\alpha = 3$,
  - $SK_7(sk1,r)float,20 \to 0.0068964092695284835701\ldots$,
    it is well approximated because $U7(sk1,r) = 6.888 \cdot 10^{-29}$;
  - $SK_7(sk2,r)float,20 \to 0.0077613102362227910095\ldots$,
    it is well approximated because $U7(sk2,r) = 1.05 \cdot 10^{-28}$;
  - $SK_7(sk3,r)float,20 \to 0.00094342579874230766729\ldots$,
    it is well approximated because $U7(sk3,r) = 1.378 \cdot 10^{-28}$.

8. Series

$$SK_8(r) = \sum_{k=r}^{\infty} \frac{sk_k}{(k-r)!}\ . \tag{6.36}$$

Program 6.9. for the approximation of $SK_8(r)$.

$$SK_8(sk,r) := \left| \begin{array}{l} SK \leftarrow 0 \\ for\ k \in r..last(sk) \\ \quad SK \leftarrow SK + \dfrac{sk_k}{(k-r)!}\ \ if\ sk_k \neq -1 \\ return\ \ SK \end{array} \right.$$

We define a function that is value the last term of the series (6.36)

$$U8(sk,r) := \frac{sk_{last(sk)}}{(last(sk)-r)!}\ .$$

Thus is obtained:

- Case $r = 1$
  - $SK_8(sk1,r)float,20 \to 5.2585108358721020744\ldots$,
    it is well approximated because $U8(sk1,r) = 3.385 \cdot 10^{-23}$;
  - $SK_8(sk2,r)float,20 \to 10.245251373119774594\ldots$,
    it is well approximated because $U8(sk2,r) = 5.1576 \cdot 10^{-23}$;
  - $SK_8(sk3,r)float,20 \to 8.9677818084073938106\ldots$,
    it is well approximated because $U7(sk3,r) = 6.769 \cdot 10^{-23}$;
- Case $r = 2$
  - $SK_8(sk1,r)float,20 \to 8.052817310007447497\ldots$,
    it is well approximated because $U8(sk1,r) = 8.123 \cdot 10^{-22}$;
  - $SK_8(sk2,r)float,20 \to 12.414832179662003511\ldots$,
    it is well approximated because $U7(sk2,r) = 1.24 \cdot 10^{-21}$;
  - $SK_8(sk3,r)float,20 \to 9.3374979438253485524\ldots$,
    it is well approximated because $U7(sk3,r) = 1.625 \cdot 10^{-21}$;

- Case $\alpha=3$,
  - $SK_8(sk1,r)\mathit{float},20\to 13.276751543252094323\ldots$,
    it is well approximated because $U8(sk1,r)=1.868\cdot 10^{-20}$;
  - $SK_8(sk2,r)\mathit{float},20\to 10.795894008560432223\ldots$,
    it is well approximated because $U8(sk2,r)=2.847\cdot 10^{-20}$;
  - $SK_8(sk3,r)\mathit{float},20\to 8.7740584270103739767$,
    it is well approximated because $U8(sk3,r)=3.737\cdot 10^{-20}$.

9. Series
$$SK_9=\sum_{k=1}^{\infty}\frac{1}{\sum\limits_{j=1}^{k} sk_j!}\ . \tag{6.37}$$

Program 6.10. for the approximation of SK_9.

$$SK_9(sk):=\left|\begin{array}{l} SK\leftarrow 0\\ for\ \ k\in 1..last(sk)\\ \quad if\ \ sk_k\neq -1\\ \quad\left|\begin{array}{l} sum\leftarrow 0\\ for\ \ j\in 1..k\\ \quad sum\leftarrow sum+sk_j!\quad if\quad sk_j\neq -1\\ SK\leftarrow SK+\dfrac{1}{sum}\end{array}\right.\\ return\ \ SK\end{array}\right.$$

Program 6.11. that is value the last term of the series (6.37).

$$U9(sk):=\left|\begin{array}{l} sum\leftarrow 0\\ for\ \ j\in 1..last(sk)\\ \quad sum\leftarrow sum+sk_j!\quad if\quad sk_j\neq -1\\ return\ \ \dfrac{1}{sum}\end{array}\right.$$

Thus is obtained:

- $SK_9(sk1)\mathit{float},20\to 0.54135130818666812662\ldots$,
  it is well approximated because $U9(sk1)\to 7.748\cdot 10^{-21}$;
- $SK_9(sk2)\mathit{float},20\to 0.51627681711181976487\ldots$,
  it is well approximated because $U9(sk2)\to 6.204\cdot 10^{-37}$;

- $SK_9(sk3)$*float*$,20 \to 0.50404957787232673113\ldots$,
  it is well approximated because $U9(sk3) \to 1.768 \cdot 10^{-52}$.

10. Series
$$SK_{10} = \sum_{k=1}^{\infty} \frac{1}{sk_k^{\alpha}\sqrt{sk_k!}}\ . \tag{6.38}$$

Program 6.12. for the approximation of SK_{10}.

$$SK_{10}(sk,\alpha) := \left| \begin{array}{l} SK \leftarrow 0 \\ for\ \ k \in 1..last(sk) \\ \quad SK \leftarrow SK + \dfrac{1}{sk_k^{\alpha}\sqrt{s_k!}} \ \ if\ sk_k \neq -1 \\ return\ \ SK \end{array} \right.$$

Program 6.13. that is value the last term of the series (6.38).

$$U10(sk,\alpha) := \left| \begin{array}{l} for\ \ k = last(sk)..1 \\ \quad return\ \ \dfrac{1}{sk_k^{\alpha}\sqrt{sk_k!}} \ \ if\ sk_k \neq -1 \end{array} \right.$$

Thus is obtained:

- Case $\alpha = 1$,
  - $SK_{10}(sk1,\alpha)$*float*$,12 \to 0.439292810686\ldots$,
    it is well approximated because $U10(sk1,\alpha) = 6.662 \cdot 10^{-12}$;
  - $SK_{10}(sk2,\alpha)$*float*$,20 \to 0.44373908470389981298\ldots$,
    it is well approximated because $U10(sk2,\alpha) = 6.092 \cdot 10^{-20}$;
  - $SK_{10}(sk3,\alpha)$*float*$,20 \to 0.43171671029085234099\ldots$,
    it is well approximated because $U10(sk3,\alpha) = 6.352 \cdot 10^{-28}$;
- Case $\alpha = 2$,
  - $SK_{10}(sk1,\alpha)$*float*$,13 \to 0.1958316244233\ldots$,
    it is well approximated because $U10(sk1,\alpha) = 3.172 \cdot 10^{-13}$;
  - $SK_{10}(sk2,\alpha)$*float*$,20 \to 0.19720311371907892905\ldots$,
    it is well approximated because $U10(sk2,\alpha) = 1.904 \cdot 10^{-21}$;
  - $SK_{10}(sk3,\alpha)$*float*$,20 \to 0.19458905271804637084\ldots$,
    it is well approximated because $U10(sk3,\alpha) = 1.512 \cdot 10^{-29}$;
- Case $\alpha = 3$,

- $SK_{10}(sk1,\alpha)float,14 \to 0.09273531642709\ldots$,
  it is well approximated because $U10(sk1,\alpha) = 1.511\cdot10^{-14}$;
- $SK_{10}(sk2,\alpha)float,20 \to 0.09308920795219201976\,5\ldots$,
  it is well approximated because $U10(sk2,\alpha) = 5.949\cdot10^{-23}$;
- $SK_{10}(sk3,\alpha)float,20 \to 0.09253165167592996270\,3\ldots$,
  it is well approximated because $U10(sk3,\alpha) = 3.601\cdot10^{-31}$;

• Case $\alpha = 4$,

- $SK_{10}(sk1,\alpha)float,16 \to 0.0452068407230367\ldots$,
  it is well approximated because $U10(sk3,\alpha) = 7.194\cdot10^{-16}$;
- $SK_{10}(sk2,\alpha)float,20 \to 0.04529073773277580492\,2\ldots$,
  it is well approximated because $U10(sk3,\alpha) = 1.859\cdot10^{-24}$;
- $SK_{10}(sk3,\alpha)float,20 \to 0.04517345328638279564\,7\ldots$,
  it is well approximated because $U10(sk3,\alpha) = 8.574\cdot10^{-33}$.

11. Series
$$SK_{11} = \sum_{k=1}^{\infty} \frac{1}{sk_k^{\alpha}\sqrt{(sk_k+1)!}}\ . \tag{6.39}$$

Program 6.14. for the approximation of SK_{11}.

$$SK_{11}(sk,\alpha) := \left| \begin{array}{l} SK \leftarrow 0 \\ for\ k \in 1..last(sk) \\ \quad SK \leftarrow SK + \dfrac{1}{sk_k^{\alpha}\sqrt{(s_k+1)!}}\ \ if\ sk_k \neq -1 \\ return\ \ SK \end{array} \right.$$

Program 6.15. that is value the last term of the series (6.38).

$$U11(sk,\alpha) := \left| \begin{array}{l} for\ k = last(sk)..1 \\ \quad return\ \dfrac{1}{sk_k^{\alpha}\sqrt{(sk_k+1)!}}\ \ if\ sk_k \neq -1 \end{array} \right.$$

Thus is obtained:

• Case $\alpha = 1$,

- $SK_{11}(sk1,\alpha)float,12 \to 0.240518730353\ldots$,
  it is well approximated because $U11(sk1,\alpha) = 6.662\cdot10^{-12}$;
- $SK_{11}(sk2,\alpha)float,20 \to 0.24277337011690480832\ldots$,
  it is well approximated because $U11(sk2,\alpha) = 6.092\cdot10^{-20}$;

  - $SK_{11}(sk3,\alpha)$*float*$,20 \rightarrow 0.23767438713448743589\ldots$,
    it is well approximated because $U11(sk3,\alpha) = 6.352\cdot 10^{-28}$;
- Case $\alpha = 2$,
  - $SK_{11}(sk1,\alpha)$*float*$,13 \rightarrow 0.1102441365259\ldots$,
    it is well approximated because $U11(sk1,\alpha) = 3.172\cdot 10^{-13}$;
  - $SK_{11}(sk2,\alpha)$*float*$,20 \rightarrow 0.11087662573522063082\ldots$,
    it is well approximated because $U11(sk2,\alpha) = 1.904\cdot 10^{-21}$;
  - $SK_{11}(sk3,\alpha)$*float*$,20 \rightarrow 0.10977783145765213226\ldots$,
    it is well approximated because $U11(sk3,\alpha) = 1.512\cdot 10^{-29}$;
- Case $\alpha = 3$,
  - $SK_{11}(sk1,\alpha)$*float*$,14 \rightarrow 0.05291501051360\ldots$,
    it is well approximated because $U11(sk1,\alpha) = 1.511\cdot 10^{-14}$;
  - $SK_{11}(sk2,\alpha)$*float*$,20 \rightarrow 0.05307145269339964 2756\ldots$,
    it is well approximated because $U11(sk2,\alpha) = 5.949\cdot 10^{-23}$;
  - $SK_{11}(sk3,\alpha)$*float*$,20 \rightarrow 0.05283858227134606 6972\ldots$,
    it is well approximated because $U11(sk3,\alpha) = 3.601\cdot 10^{-}31$;
- Case $\alpha = 4$,
  - $SK_{11}(sk1,\alpha)$*float*$,16 \rightarrow 0.0259576233294432\ldots$,
    it is well approximated because $U11(sk3,\alpha) = 7.194\cdot 10^{-16}$;
  - $SK_{11}(sk2,\alpha)$*float*$,20 \rightarrow 0.02599384554099203 7418\ldots$,
    it is well approximated because $U11(sk3,\alpha) = 1.859\cdot 10^{-24}$;
  - $SK_{11}(sk3,\alpha)$*float*$,20 \rightarrow 0.02594509130346593 4837\ldots$,
    it is well approximated because $U11(sk3,\alpha) = 8.574\cdot 10^{-33}$.

6.4 Smarandache–Wagstaff Constants

The authors did not prove the convergence towards each constant. We let it as possible research for the interested readers. With the program *SW* 2.82 calculate vectors top 25 terms numbers containing Smarandache–Wagstaff, *sw1* = *SW*(1, *p*), *sw2* = *SW*(2, *p*) and *sw3* = *SW*(3, *p*), where

$$p = (2\ 3\ 5\ 7\ 11\ 13\ 17\ 19\ 23\ 29\ 31\ 37\ 41\ 43\ 47\ 53\ 59\ 61\ 67\ 71\ 73\ 79\ 83\ 89\ 97)^{\mathrm{T}}\ .$$

Vectors *sw1* (2.98), *sw2* (2.99) and *sw3* (2.100) has 25 terms.

In a similar manner with Smarandache–Kurepa constants were obtained Smarandache–Wagstaff constants, which are found in following table.

Table 6.2: Smarandache–Wagstaff constants

Name	Constant value	Value the last term
$SW_1(sw1)$	0.55158730367497730335...	$1.389\cdot10^{-3}$
$SW_1(sw2)$	0.71825397034164395277...	$1.216\cdot10^{-34}$
$SW_1(sw3)$	0.70994819257251365270...	$2.9893\cdot10^{-50}$
$SW_2(sw1)$	1.0343637569611291909...	$3.868\cdot10^{-25}$
$SW_2(sw2)$	4.2267823464172704922...	$1.999\cdot10^{-24}$
$SW_2(sw3)$	5.1273356604617316278...	$2.643\cdot10^{-24}$
$SW_3(sw1)$	0.65219770185345831168...	$6.208\cdot10^{-23}$
$SW_3(sw2)$	0.54968878346863715478...	$1.985\cdot10^{-30}$
$SW_3(sw3)$	0.54699912527156976558...	$6.180\cdot10^{-32}$
$SW_4(sw1,1)$	1.8199032834559367993...	$1.552\cdot10^{-21}$
$SW_4(sw2,1)$	0.87469626917369886975...	$4.963\cdot10^{-29}$
$SW_4(sw3,1)$	0.81374377609424443467...	$1.545\cdot10^{-30}$
$SW_4(sw1,2)$	6.5303985125207189262...	$3.880\cdot10^{-20}$
$SW_4(sw2,2)$	1.8815309698588492643...	$1.241\cdot10^{-27}$
$SW_4(sw3,2)$	1.4671407531614048561...	$3.862\cdot10^{-29}$
$SW_4(sw1,3)$	29.836629842971767949...	$9.700\cdot10^{-19}$
$SW_4(sw2,3)$	5.4602870746051287154...	$3.102\cdot10^{-26}$
$SW_4(sw3,3)$	3.1799919620918477289...	$9.656\cdot10^{-28}$
$SW_4(sw1,4)$	161.01437206742466933...	$2.425\cdot10^{-17}$
$SW_4(sw2,4)$	19.652380984010238861...	$7.755\cdot10^{-25}$
$SW_4(sw3,4)$	8.0309117554069796125...	$2.414\cdot10^{-26}$
$SW_5(sw1)$	−0.96564681546640958928...	$3.868\cdot10^{-25}$
$SW_5(sw2)$	1.8734063576918609871...	$1.999\cdot10^{-24}$
$SW_5(sw3)$	2.3298081028442529042...	$2.643\cdot10^{-24}$
$SW_6(sw1)$	0.33901668392958325553...	$1.488\cdot10^{-26}$
$SW_6(sw2)$	1.8764315154663871518...	$7.687\cdot10^{-26}$
$SW_6(sw3)$	2.0885585895284377234...	$1.017\cdot10^{-25}$
$SW_7(sw1,1)$	0.33901668392958325553...	$1.48\cdot10^{-26}$
$SW_7(sw2,1)$	1.8764315154663871518...	$7.687\cdot10^{-26}$
$SW_7(sw3,1)$	2.0885585895284377234...	$1.0167\cdot10^{-25}$
$SW_7(sw1,2)$	0.084141103378415065393...	$5.510\cdot10^{-28}$
$SW_7(sw2,2)$	0.090257150078331834732...	$2.847\cdot10^{-27}$
$SW_7(sw3,2)$	0.13102847128229505811...	$3.765\cdot10^{-27}$
$SW_7(sw1,3)$	0.00010061233843047075978...	$1.968\cdot10^{-29}$
$SW_7(sw2,3)$	0.0009621231898758737862...	$1.017\cdot10^{-28}$
$SW_7(sw3,3)$	0.0075668414805812168739...	$1.345\cdot10^{-28}$

Continued on next page

Name	Constant value	Value the last term
$SW_8(sw1,1)$	2.173961760187995128...	$9.670\cdot 10^{-24}$
$SW_8(sw2,1)$	5.9782465282928175363...	$4.996\cdot 10^{-23}$
$SW_8(sw3,1)$	8.9595673801550634215...	$6.608\cdot 10^{-23}$
$SW_8(sw1,2)$	2.7111929711924074628...	$2.320\cdot 10^{-22}$
$SW_8(sw2,2)$	5.315290260964323833...	$1.199\cdot 10^{-21}$
$SW_8(sw3,2)$	12.533730523303497135...	$1.586\cdot 10^{-21}$
$SW_8(sw1,3)$	2.2288452763809560596...	$5.338\cdot 10^{-21}$
$SW_8(sw2,3)$	8.3156791068197439503...	$2.758\cdot 10^{-20}$
$SW_8(sw3,3)$	20.136561600709999638...	$3.648\cdot 10^{-20}$
$SW_9(sw1)$	0.54531085561770668291...	$5.872\cdot 10^{-19}$
$SW_9(sw2)$	0.32441910792206133261...	$3.028\cdot 10^{-36}$
$SW_9(sw3)$	0.32292213990139703594...	$4.921\cdot 10^{-51}$
$SW_{10}(sw1,1)$	0.4310692254141283029...	$6.211\cdot 10^{-3}$
$SW_{10}(sw2,1)$	0.56715198854304553087...	$3.557\cdot 10^{-19}$
$SW_{10}(sw3,1)$	0.54969992215171005647...	$4.217\cdot 10^{-27}$
$SW_{10}(sw1,2)$	0.19450893949849271072...	$1.035\cdot 10^{-3}$
$SW_{10}(sw2,2)$	0.23986986064238086601...	$1.148\cdot 10^{-20}$
$SW_{10}(sw3,2)$	0.23631653581644449211...	$1.029\cdot 10^{-28}$
$SW_{10}(sw1,3)$	0.092521713489107819791...	$1.726\cdot 10^{-4}$
$SW_{10}(sw2,3)$	0.107642020542344188...	$3.702\cdot 10^{-22}$
$SW_{10}(sw3,3)$	0.1069237140845950118...	$2.509\cdot 10^{-30}$
$SW_{10}(sw1,4)$	0.045172218019205576526...	$2.876\cdot 10^{-5}$
$SW_{10}(sw2,4)$	0.050212320370560015101...	$1.194\cdot 10^{-23}$
$SW_{10}(sw3,4)$	0.050067728660537151051...	$6.119\cdot 10^{-32}$
$SW_{11}(sw1,1)$	0.23745963081186272553...	$6.211\cdot 10^{-3}$
$SW_{11}(sw2,1)$	0.30550101247582884341...	$3.557\cdot 10^{-19}$
$SW_{11}(sw3,1)$	0.29831281390164911686...	$4.217\cdot 10^{-27}$
$SW_{11}(sw1,2)$	0.10975122611929470037...	$1.035\cdot 10^{-3}$
$SW_{11}(sw2,2)$	0.13243168669643861815...	$1.148\cdot 10^{-20}$
$SW_{11}(sw3,2)$	0.13097335030519627439...	$1.029\cdot 10^{-28}$
$SW_{11}(sw1,3)$	0.052835278683252704210...	$1.726\cdot 10^{-4}$
$SW_{11}(sw2,3)$	0.060395432210142668039...	$3.702\cdot 10^{-22}$
$SW_{11}(sw3,3)$	0.060101248246962299788...	$2.509\cdot 10^{-30}$
$SW_{11}(sw1,4)$	0.025944680389944764723...	$2.876\cdot 10^{-5}$
$SW_{11}(sw2,4)$	0.028464731565636192167...	$1.194\cdot 10^{-23}$
$SW_{11}(sw3,4)$	0.028405588029145682932...	$6.119\cdot 10^{-32}$

6.5 Smarandache Ceil Constants

The authors did not prove the convergence towards each constant. We let it as possible research for the interested readers. With the program *Sk* 2.92 calculate vectors top 100 terms numbers containing Smarandache Ceil, $sk1 := Sk(100,1)$, $sk2 := Sk(100,2)$, $sk3 := Sk(100,3)$, $sk4 := Sk(100,4)$, $sk5 := Sk(100,5)$ and $sk6 = Sk(100,6)$ given on page 90.

$$Sk_1(sk) := \sum_{k=1}^{last(sk)} \frac{1}{sk_k!} \approx \sum_{k=1}^{\infty} \frac{1}{sk_k!}\,. \tag{6.40}$$

$$Sk_2(sk) := \sum_{k=1}^{last(sk)} \frac{sk_k}{k!} \approx \sum_{k=1}^{\infty} \frac{sk_k}{k!}\,. \tag{6.41}$$

$$Sk_3(sk) := \sum_{k=1}^{last(sk)} \frac{1}{\prod_{j=1}^{k} sk_j} \approx \sum_{k=1}^{\infty} \frac{1}{\prod_{j=1}^{k} sk_j}\,. \tag{6.42}$$

$$Sk_4(sk,\alpha) := \sum_{k=1}^{last(sk)} \frac{k^\alpha}{\prod_{j=1}^{k} sk_j} \approx \sum_{k=1}^{\infty} \frac{k^\alpha}{\prod_{j=1}^{k} sk_j}\,. \tag{6.43}$$

$$Sk_5(sk) := \sum_{k=1}^{last(sk)} \frac{(-1)^{k+1} sk_k}{k!} \approx \sum_{k=1}^{\infty} \frac{(-1)^{k+1} sk_k}{k!}\,. \tag{6.44}$$

$$Sk_6(sk) := \sum_{k=1}^{last(sk)} \frac{sk_k}{(k+1)!} \approx \sum_{k=1}^{\infty} \frac{sk_k}{(k+1)!}\,. \tag{6.45}$$

$$Sk_7(sk,r) := \sum_{k=r}^{last(sk)} \frac{sk_k}{(k+r)!} \approx \sum_{k=r}^{\infty} \frac{sk_k}{(k+r)!}\,. \tag{6.46}$$

$$Sk_8(sk,r) := \sum_{k=r}^{last(sk)} \frac{sk_k}{(k-r)!} \approx \sum_{k=r}^{\infty} \frac{sk_k}{(k-r)!}\,. \tag{6.47}$$

$$Sk_9(sk) := \sum_{k=1}^{last(sk)} \frac{1}{\sum_{j=1}^{k} sk_j!} \approx \sum_{k=1}^{\infty} \frac{1}{\sum_{j=1}^{k} sk_j!}\,. \tag{6.48}$$

$$Sk_{10}(sk,\alpha) := \sum_{k=1}^{last(sk)} \frac{1}{sk_k^\alpha \sqrt{sk_k!}} \approx \sum_{k=1}^{\infty} \frac{1}{sk_k^\alpha \sqrt{sk_k!}}\,. \tag{6.49}$$

$$Sk_{11}(sk,\alpha) := \sum_{k=1}^{last(sk)} \frac{1}{sk_k^{\alpha}\sqrt{(sk_k+1)!}} \approx \sum_{k=1}^{\infty} \frac{1}{sk_k^{\alpha}\sqrt{(sk_k+1)!}}\ . \tag{6.50}$$

In the formulas (6.40–6.50) will replace sk with $sk1$ or $sk2$ or $\ldots sk6$.

Table 6.3: Smarandache ceil constants

Name	Constant value
$Sk_1(sk1)$	1.7182818284590452354
$Sk_1(sk2)$	2.4393419627293664997
$Sk_1(sk3)$	3.1517898773198280566
$Sk_1(sk4)$	3.7781762839623110876
$Sk_1(sk5)$	4.2378985040968576110
$Sk_1(sk6)$	4.6962318374301909444
$Sk_2(sk1)$	2.7182818284590452354
$Sk_2(sk2)$	2.6348327418583415529
$Sk_2(sk3)$	2.6347831386837383783
$Sk_2(sk4)$	2.6347831386836427887
$Sk_2(sk5)$	2.6347831386836427887
$Sk_2(sk6)$	2.6347831386836427887
$Sk_3(sk1)$	1.7182818284590452354
$Sk_3(sk2)$	1.7699772067340966537
$Sk_3(sk3)$	1.7701131436269234662
$Sk_3(sk4)$	1.7701131436367350613
$Sk_3(sk5)$	1.7701131436367350613
$Sk_3(sk6)$	1.7701131436367350613
$Sk_4(sk1,1)$	2.7182818284590452354
$Sk_4(sk2,1)$	2.9372394121769037872
$Sk_4(sk3,1)$	2.9383677132426964633
$Sk_4(sk4,1)$	2.9383677134004179370
$Sk_4(sk5,1)$	2.9383677134004179370
$Sk_4(sk6,1)$	2.9383677134004179370
$Sk_4(sk1,2)$	5.4365636569180904707
$Sk_4(sk2,2)$	6.3788472114323090813
$Sk_4(sk3,2)$	6.3882499784201737207
$Sk_4(sk4,2)$	6.3882499809564429861
$Sk_4(sk5,2)$	6.3882499809564429861
$Sk_4(sk6,2)$	6.3882499809564429861

Continued on next page

Name	Constant value
$Sk_4(sk1,3)$	13.591409142295226177
$Sk_4(sk2,3)$	17.731481467469518346
$Sk_4(sk3,3)$	17.810185157161258977
$Sk_4(sk4,3)$	17.810185197961806356
$Sk_4(sk5,3)$	17.810185197961806356
$Sk_4(sk6,3)$	17.810185197961806356
$Sk_5(sk1)$	0.3678794411714423216
$Sk_5(sk2)$	0.4512954589890772210 2
$Sk_5(sk3)$	0.45134506216368039562
$Sk_5(sk4)$	0.45134506216377598517
$Sk_5(sk5)$	0.45134506216377598517
$Sk_5(sk6)$	0.45134506216377598517
$Sk_6(sk1)$	1.0
$Sk_6(sk2)$	0.98332065600291383328
$Sk_6(sk3)$	0.98331514453906903611
$Sk_6(sk4)$	0.98331514453906341319
$Sk_6(sk5)$	0.98331514453906341319
$Sk_6(sk6)$	0.98331514453906341319
$Sk_7(sk1,2)$[1]	0.11505150487428809797
$Sk_7(sk2,2)$	0.11227247442226469528
$Sk_7(sk3,2)$	0.11227192327588021556
$Sk_7(sk4,2)$	0.11227192327587990317
$Sk_7(sk5,2)$	0.11227192327587990317
$Sk_7(sk6,2)$	0.11227192327587990317
$Sk_7(sk1,3)$	0.0051030097485761959461
$Sk_7(sk2,3)$	0.0047060716126746675110
$Sk_7(sk3,3)$	0.0047060215084578966275
$Sk_7(sk4,3)$	0.0047060215084578801863
$Sk_7(sk5,3)$	0.0047060215084578801863
$Sk_7(sk6,3)$	0.0047060215084578801863
$Sk_7(sk1,4)$	0.00011483208318175423660 0
$Sk_7(sk2,4)$	0.000065219594046380693870
$Sk_7(sk3,4)$	0.000065215418694983120250
$Sk_7(sk4,4)$	0.000065215418694982298187
$Sk_7(sk5,4)$	0.000065215418694982298187
$Sk_7(sk6,4)$	0.000065215418694982298187
$Sk_8(sk1,1)$	5.4365636569180904707

Continued on next page

[1] $SK_7(sk1,1) = SK_6(sk1)\ldots SK_7(sk6,1) = SK_6(sk6)$

Name	Constant value
$Sk_8(sk2,1)$	5.1022877129454360911
$Sk_8(sk3,1)$	5.1018908875486106943
$Sk_8(sk4,1)$	5.1018908875470812615
$Sk_8(sk5,1)$	5.1018908875470812615
$Sk_8(sk6,1)$	5.1018908875470812615
$Sk_8(sk1,2)$	8.1548454853771357061
$Sk_8(sk2,2)$	7.1480978000537264745
$Sk_8(sk3,2)$	7.1453200222759486967
$Sk_8(sk4,2)$	7.1453200222530072055
$Sk_8(sk5,2)$	7.1453200222530072055
$Sk_8(sk6,2)$	7.1453200222530072055
$Sk_8(sk1,3)$	10.873127313836180941
$Sk_8(sk2,3)$	8.8314441108416899115
$Sk_8(sk3,3)$	8.8147774441750232447
$Sk_8(sk4,3)$	8.8147774438538423680
$Sk_8(sk5,3)$	8.8147774438538423680
$Sk_8(sk6,3)$	8.8147774438538423680
$Sk_9(sk1)$	1.4826223630822915238
$Sk_9(sk2)$	1.5446702350540098246
$Sk_9(sk3)$	1.5446714960716397966
$Sk_9(sk4)$	1.5446714960716397966
$Sk_9(sk5)$	1.5446714960716397966
$Sk_9(sk6)$	1.5446714960716397966
$Sk_{10}(sk1,1)$	1.5680271290107037107
$Sk_{10}(sk2,1)$	2.1485705607791708605
$Sk_{10}(sk3,1)$	2.7064911009876044790
$Sk_{10}(sk4,1)$	3.1511717572816964677
$Sk_{10}(sk5,1)$	3.4599016039136594552
$Sk_{10}(sk6,1)$	3.7624239581989503403
$Sk_{10}(sk1,2)$	1.2399748241535239012
$Sk_{10}(sk2,2)$	1.4820295340881653857
$Sk_{10}(sk3,2)$	1.7198593859973659544
$Sk_{10}(sk4,2)$	1.9302588979239212805
$Sk_{10}(sk5,2)$	2.0953127335005767356
$Sk_{10}(sk6,2)$	2.2593316697202178974
$Sk_{10}(sk1,3)$	1.1076546756800267505
$Sk_{10}(sk2,3)$	1.2156553766098991247
$Sk_{10}(sk3,3)$	1.3228498484138567230
$Sk_{10}(sk4,3)$	1.4233398208190790660

Continued on next page

Name	Constant value
$Sk_{10}(sk5,3)$	1.5087112383659205930
$Sk_{10}(sk6,3)$	1.5939101462449901038
$Sk_{11}(sk1,1)$	1.0128939498871834093
$Sk_{11}(sk2,1)$	1.3234047869734872073
$Sk_{11}(sk3,1)$	1.6249765356359900199
$Sk_{11}(sk4,1)$	1.8766243840544798818
$Sk_{11}(sk5,1)$	2.0602733507289191532
$Sk_{11}(sk6,1)$	2.2415757227314687400
$Sk_{11}(sk1,2)$	0.83957287300941603913
$Sk_{11}(sk2,2)$	0.97282437595501178612
$Sk_{11}(sk3,2)$	1.10438931993466493290
$Sk_{11}(sk4,2)$	1.22381269885269296020
$Sk_{11}(sk5,2)$	1.32056051527626897540
$Sk_{11}(sk6,2)$	1.41691714458488924910
$Sk_{11}(sk1,3)$	0.76750637030682730577
$Sk_{11}(sk2,3)$	0.82803667284162935790
$Sk_{11}(sk3,3)$	0.88824280769856038385
$Sk_{11}(sk4,3)$	0.94547228007452707047
$Sk_{11}(sk5,3)$	0.99514216074174737828
$Sk_{11}(sk6,3)$	1.04474683622289388520

6.6 Smarandache–Mersenne Constants

The authors did not prove the convergence towards each constant. We let it as possible research for the interested readers. With the program *SML* 2.94 and *SMR* 2.96 calculate vectors top 40 terms numbers containing Smarandache–Mersenne, $n := 1..40$, $smlp := SML(prime_n)$, $sml\omega := SML(2n-1)$, given on page 92 and $smrp := SMR(prime_n)$, $smr\omega := SMR(2n-1)$, given on page 92. Using programs similar to programs *SK1* 6.2–*SK11* 6.14 was calculated constants Smarandache–Mersenne.

$$SM_1(sm) = \sum_{k=1}^{last(sm)} \frac{1}{sm_k!}\,. \tag{6.51}$$

$$SM_2(sm) = \sum_{k=1}^{last(sm)} \frac{sm_k}{k!}\,. \tag{6.52}$$

$$SM_3(sm)=\sum_{k=1}^{last(sm)}\frac{1}{\prod_{j=1}^{k}sm_j}\ . \tag{6.53}$$

$$SM_4(sm,\alpha)=\sum_{k=1}^{last(sm)}\frac{k^\alpha}{\prod_{j=1}^{k}sm_j}\ ,\ \text{ where }\ \alpha\in\mathbb{N}^*\ . \tag{6.54}$$

$$SM_5(sm)=\sum_{k=1}^{last(sm)}\frac{(-1)^{k+1}sm_k}{k!} \tag{6.55}$$

$$SM_6(sm)=\sum_{k=1}^{last(sm)}\frac{sm_k}{(k+1)!} \tag{6.56}$$

$$SM_7(sm,r)=\sum_{k=r}^{last(sm)}\frac{sm_k}{(k+r)!}\ ,\ \text{ where }\ r\in\mathbb{N}^*\ . \tag{6.57}$$

$$SM_8(sm,r)=\sum_{k=r}^{last(sm)}\frac{sm_k}{(k-r)!}\ ,\ \text{ where }\ r\in\mathbb{N}^*\ . \tag{6.58}$$

$$SM_9(sm)=\sum_{k=1}^{last(sm)}\frac{1}{\sum_{j=1}^{k}sm_k!} \tag{6.59}$$

$$SM_{10}(sm,\alpha)=\sum_{k=1}^{last(sm)}\frac{1}{sm_k^\alpha\sqrt{sm_k!}}\ ,\ \text{ where }\ \alpha\in\mathbb{N}^*\ . \tag{6.60}$$

$$SM_{11}(sm,\alpha)=\sum_{k=1}^{last(sm)}\frac{1}{sm_k^\alpha\sqrt{(sm_k+1)!}}\ ,\ \text{ where }\ \alpha\in\mathbb{N}^*\ . \tag{6.61}$$

In the formulas (6.51–6.61) will replace *sm* with *smlp* or *smlω* or *smrp* or *smrω*.

Table 6.4: Smarandache–Mersenne constants

Name	Constant value	Value the last term
$SM_1(smlp)$	0.71689296446162367907...	$4.254\cdot10^{-79}$
$SM_1(sml\omega)$	1.76255294555486814\81...	$4.902\cdot10^{-47}$
$SM_1(smrp)$	1.55159033291535802\93...	$4.254\cdot10^{-79}$
$SM_1(smr\omega)$	2.72798500882981305\35...	$1.406\cdot10^{-75}$
$SM_2(smlp)$	1.89373862973543901\32...	$7.109\cdot10^{-47}$
$SM_2(sml\omega)$	2.85806289717560030\18...	$4.780\cdot10^{-47}$
$SM_2(smrp)$	0.88435409570308110516...	$7.106\cdot10^{-47}$
$SM_2(smr\omega)$	1.86648175877453505\01...	$6.863\cdot10^{-47}$

Continued on next page

Name	Constant value	Value the last term
$SM_3(smlp)$	0.67122659824508457394...	$5.222\cdot 10^{-53}$
$SM_3(sml\omega)$	1.6743798207846741489...	$6.111\cdot 10^{-44}$
$SM_3(smrp)$	1.6213314132399371377...	$8.770\cdot 10^{-49}$
$SM_3(smr\omega)$	2.7071160606980893623...	$2.243\cdot 10^{-31}$
$SM_4(smlp,1)$	1.5649085118184007572...	$2.089\cdot 10^{-51}$
$SM_4(sml\omega,1)$	2.5810938731370101206...	$2.444\cdot 10^{-42}$
$SM_4(smrp,1)$	4.1332258765524751774...	$3.508\cdot 10^{-47}$
$SM_4(smr\omega,1)$	5.5864923376104430121...	$8.974\cdot 10^{-30}$
$SM_4(smlp,2)$	3.9106331251114174037...	$8.356\cdot 10^{-50}$
$SM_4(sml\omega,2)$	4.9941981422394427409...	$9.777\cdot 10^{-41}$
$SM_4(smrp,2)$	11.837404185675137231...	$1.403\cdot 10^{-45}$
$SM_4(smr\omega,2)$	15.26980028696103804...	$3.590\cdot 10^{-28}$
$SM_4(smlp,3)$	10.653791515005141146...	$3.342\cdot 10^{-48}$
$SM_4(sml\omega,3)$	12.088423092839839475...	$3.911\cdot 10^{-39}$
$SM_4(smrp,3)$	39.302927041913622498...	$5.613\cdot 10^{-44}$
$SM_4(smr\omega,3)$	53.644863796864365278...	$1.436\cdot 10^{-26}$
$SM_5(smlp)$	–0.3905031434783375128...	$-7.109\cdot 10^{-47}$
$SM_5(sml\omega)$	0.58007673972219526962...	$-4.780\cdot 10^{-47}$
$SM_5(smrp)$	–0.13276679090380714341...	$-7.109\cdot 10^{-47}$
$SM_5(smr\omega)$	0.85258790936171242297...	$-6.863\cdot 10^{-47}$
$SM_6(smlp)$	0.54152160636801684581...	$1.734\cdot 10^{-48}$
$SM_6(sml\omega)$	1.0356287723533406243...	$1.166\cdot 10^{-48}$
$SM_6(smrp)$	0.25825928231171221647...	$1.734\cdot 10^{-48}$
$SM_6(smr\omega)$	0.75530886756654392948...	$1.674\cdot 10^{-48}$
$SM_7(smlp,2)$	0.12314242079796645308...	$4.128\cdot 10^{-50}$
$SM_7(sml\omega,2)$	0.12230623557364785581...	$2.776\cdot 10^{-50}$
$SM_7(smrp,2)$	0.059487738877545345581...	$4.128\cdot 10^{-50}$
$SM_7(smr\omega,2)$	0.059069232762245306011...	$3.986\cdot 10^{-50}$
$SM_7(smlp,3)$	0.0064345613772149429216...	$9.600\cdot 10^{-52}$
$SM_7(sml\omega,3)$	0.0063305872508483357372...	$6.455\cdot 10^{-52}$
$SM_7(smrp,3)$	0.0029196501300934203105...	$9.600\cdot 10^{-52}$
$SM_7(smr\omega,3)$	0.0028676244349701518512...	$9.269\cdot 10^{-52}$
$SM_8(smlp,1)$	5.0317015083535379435...	$2.843\cdot 10^{-45}$
$SM_8(sml\omega,1)$	5.8510436406486760859...	$1.912\cdot 10^{-45}$
$SM_8(smrp,1)$	2.2657136563817159215...	$2.843\cdot 10^{-45}$
$SM_8(smr\omega,1)$	3.1751240304194110976...	$2.745\cdot 10^{-45}$
$SM_8(smlp,2)$	9.7612345513619487842...	$1.109\cdot 10^{-43}$
$SM_8(sml\omega,2)$	9.0242760612601416584...	$7.457\cdot 10^{-44}$
$SM_8(smrp,2)$	4.1295191146631057347...	$1.109\cdot 10^{-43}$

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Name	Constant value	Value the last term
$SM_8(smr\omega,2)$	3.7593504699750039665...	$1.071 \cdot 10^{-43}$
$SM_8(smlp,3)$	14.504395800227427856...	$4.214 \cdot 10^{-42}$
$SM_8(sml\omega,3)$	12.21486768404280473...	$2.834 \cdot 10^{-42}$
$SM_8(smrp,3)$	5.7444969832482504712...	$4.214 \cdot 10^{-42}$
$SM_8(smr\omega,3)$	4.5906776136611764786...	$4.069 \cdot 10^{-42}$
$SM_9(smlp)$	0.56971181817608535663...	$1.937 \cdot 10^{-84}$
$SM_9(sml\omega)$	1.4020017036535156398...	$1.202 \cdot 10^{-82}$
$SM_9(smrp)$	1.3438058059670733965...	$1.970 \cdot 10^{-84}$
$SM_9(smr\omega)$	1.8600161722383592123...	$6.567 \cdot 10^{-85}$
$SM_{10}(smlp,1)$	0.56182922629094674981...	$1.125 \cdot 10^{-41}$
$SM_{10}(sml\omega,1)$	1.630157684938493886...	$1.795 \cdot 10^{-25}$
$SM_{10}(smrp,1)$	1.4313028466067383441...	$1.125 \cdot 10^{-41}$
$SM_{10}(smr\omega,1)$	2.5922754787881733157...	$6.697 \cdot 10^{-40}$
$SM_{10}(smlp,2)$	0.23894083955413619831...	$1.939 \cdot 10^{-43}$
$SM_{10}(sml\omega,2)$	1.2545998023407273118...	$4.603 \cdot 10^{-27}$
$SM_{10}(smrp,2)$	1.1945344006412840488...	$1.939 \cdot 10^{-43}$
$SM_{10}(smr\omega,2)$	2.2446282430004759916...	$1.1959 \cdot 10^{-41}$
$SM_{10}(smlp,3)$	0.10748225265502766995...	$3.343 \cdot 10^{-45}$
$SM_{10}(sml\omega,3)$	1.1111584723169680866...	$1.180 \cdot 10^{-28}$
$SM_{10}(smrp,3)$	1.0925244916164133302...	$3.343 \cdot 10^{-45}$
$SM_{10}(smr\omega,3)$	2.1085527095304115107...	$2.136 \cdot 10^{-43}$
$SM_{11}(smlp,1)$	0.30344340711709933226...	$1.464 \cdot 10^{-42}$
$SM_{11}(sml\omega,1)$	1.0399268570536074792...	$2.839 \cdot 10^{-26}$
$SM_{11}(smrp,1)$	0.9446396368342379653...	$1.464 \cdot 10^{-42}$
$SM_{11}(smr\omega,1)$	1.7297214313345287702...	$8.870 \cdot 10^{-41}$
$SM_{11}(smlp,2)$	0.13207517402450262085...	$2.524 \cdot 10^{-44}$
$SM_{11}(sml\omega,2)$	0.84598708313977015251...	$7.279 \cdot 10^{-28}$
$SM_{11}(smrp,2)$	0.81686599189767927686...	$2.524 \cdot 10^{-44}$
$SM_{11}(smr\omega,2)$	1.5485497608283970434...	$1.58 \cdot 10^{-42}$
$SM_{11}(smlp,3)$	0.060334409371851792881...	$4.352 \cdot 10^{-46}$
$SM_{11}(sml\omega,3)$	0.76905206513210718626...	$1.866 \cdot 10^{-29}$
$SM_{11}(smrp,3)$	0.75994293154985848019...	$4.352 \cdot 10^{-46}$
$SM_{11}(smr\omega,3)$	1.4749748193334142711...	$2.829 \cdot 10^{-44}$

6.7 Smarandache Near to Primorial Constants

The authors did not prove the convergence towards each constant. We let it as possible research for the interested readers. With the program *SNtkP* 2.88 calculate vectors top 45 terms numbers containing Smarandache Near to k Primorial, $n := 1..45$, $sntp := SNtkP(n,1)$, $sntdp := SNtkP(n,2)$ and $snttp := SNtkP(n,3)$ given on page 89 and by (6.62–6.64). Using programs similar to programs *SK1* – *SK11*, 6.2–6.14, was calculated constants Smarandache near to k primorial.

$$sntp := (1\ 2\ 2\ -1\ 3\ 3\ 3\ -1\ -1\ 5\ 7\ -1\ 13\ 7\ 5\ 43\ 17\ 47\ 7\ 47\ 7\ 11\ 23\ 47\ 47\ 13\ 43\ 47\ 5\ 5\ 5\ 47\ 11\ 17\ 7\ 47\ 23\ 19\ 13\ 47\ 41\ 7\ 43\ 47\ 47)^{\mathrm{T}} \quad (6.62)$$

$$sntdp := (2\ 2\ 2\ 3\ 5\ -1\ 7\ 13\ 5\ 5\ 5\ 83\ 13\ 83\ 83\ 13\ 13\ 83\ 19\ 7\ 7\ 7\ 23\ 83\ 37\ 83\ 23\ 83\ 29\ 83\ 31\ 83\ 89\ 13\ 83\ 83\ 11\ 97\ 13\ 71\ 23\ 83\ 43\ 89\ 89)^{\mathrm{T}} \quad (6.63)$$

$$snttp := (2\ 2\ 2\ 3\ 5\ 5\ 7\ 11\ 23\ 43\ 11\ 89\ 7\ 7\ 7\ 11\ 11\ 23\ 19\ 71\ 37\ 13\ 23\ 89\ 71\ 127\ 97\ 59\ 29\ 127\ 31\ 11\ 11\ 11\ 127\ 113\ 37\ 103\ 29\ 131\ 41\ 37\ 31\ 23\ 131)^{\mathrm{T}} \quad (6.64)$$

$$SNtP_1(s) = \sum_{k=1}^{last(s)} \frac{1}{s_k!}\,. \quad (6.65)$$

$$SNtP_2(s) = \sum_{k=1}^{last(s)} \frac{s_k}{k!}\,. \quad (6.66)$$

$$SNtP_3(s) = \sum_{k=1}^{last(s)} \frac{1}{\prod_{j=1}^{k} s_j}\,. \quad (6.67)$$

$$SNtP_4(s,\alpha) = \sum_{k=1}^{last(s)} \frac{k^\alpha}{\prod_{j=1}^{k} s_j}\,, \text{ where } \alpha \in \mathbb{N}^*\,. \quad (6.68)$$

$$SNtP_5(s) = \sum_{k=1}^{last(s)} \frac{(-1)^{k+1} s_k}{k!} \tag{6.69}$$

$$SNtP_6(s) = \sum_{k=1}^{last(s)} \frac{s_k}{(k+1)!} \tag{6.70}$$

$$SNtP_7(s,r) = \sum_{k=r}^{last(s)} \frac{s_k}{(k+r)!}\ , \text{ where } r \in \mathbb{N}^*\ . \tag{6.71}$$

$$SNtP_8(s,r) = \sum_{k=r}^{last(s)} \frac{s_k}{(k-r)!}\ , \text{ where } r \in \mathbb{N}^*\ . \tag{6.72}$$

$$SNtP_9(s) = \sum_{k=1}^{last(s)} \frac{1}{\sum_{j=1}^{k} s_k!} \tag{6.73}$$

$$SNtP_{10}(s,\alpha) = \sum_{k=1}^{last(s)} \frac{1}{s_k^{\alpha}\sqrt{s_k!}}\ , \text{ where } \alpha \in \mathbb{N}^*\ . \tag{6.74}$$

$$SNtP_{11}(s,\alpha) = \sum_{k=1}^{last(s)} \frac{1}{s_k^{\alpha}\sqrt{(s_k+1)!}}\ , \text{ where } \alpha \in \mathbb{N}^*\ . \tag{6.75}$$

In the formulas (6.65–6.75) will replace s with *sntp* or *sntdp* or *snttp*.

Table 6.5: Smarandache near to k primorial constants

Name	Constant value	Value the last term
$SNtP_1(sntp)$	2.5428571934431365743...	$3.867 \cdot 10^{-60}$
$SNtP_1(sntdp)$	1.7007936768093018175...	$6.058 \cdot 10^{-137}$
$SNtP_1(snttp)$	1.6841271596523332717...	$1.180 \cdot 10^{-222}$
$SNtP_2(sntp)$	2.3630967934998628805...	$3.929 \cdot 10^{-55}$
$SNtP_2(sntdp)$	3.5017267676908976891...	$7.440 \cdot 10^{-55}$
$SNtP_2(snttp)$	3.5086818448618034565...	$1.095 \cdot 10^{-54}$
$SNtP_3(sntp)$	1.8725106254184368208...	$1.902 \cdot 10^{-46}$
$SNtP_3(sntdp)$	0.92630477141852643895...	$3.048 \cdot 10^{-60}$
$SNtP_3(snttp)$	0.92692737191576132359...	$7.637 \cdot 10^{-62}$
$SNtP_4(sntp,1)$	3.4198909215383519666...	$8.560 \cdot 10^{-45}$
$SNtP_4(sntdp,1)$	1.5926089060696178054...	$1.371 \cdot 10^{-58}$
$SNtP_4(snttp,1)$	1.5951818710249031148...	$3.437 \cdot 10^{-60}$

Continued on next page

Name	Constant value	Value the last term
$SNtP_4(sntp,2)$	9.0083762775802033621...	$3.852\cdot10^{-43}$
$SNtP_4(sntdp,2)$	3.5661339881230530766...	$6.171\cdot10^{-57}$
$SNtP_4(snttp,2)$	3.5731306093562134618...	$1.547\cdot10^{-58}$
$SNtP_4(sntp,3)$	33.601268780655137928...	$1.733\cdot10^{-41}$
$SNtP_4(sntdp,3)$	10.05655962385479256...	$2.777\cdot10^{-55}$
$SNtP_4(snttp,3)$	10.036792885247337658...	$6.959\cdot10^{-57}$
$SNtP_5(sntp)$	0.35476070426989163902...	$3.929\cdot10^{-55}$
$SNtP_5(sntdp)$	1.2510788222295556551...	$7.440\cdot10^{-55}$
$SNtP_5(snttp)$	1.2442232499898234684...	$1.095\cdot10^{-54}$
$SNtP_6(sntp)$	0.92150311621958362854...	$8.541\cdot10^{-57}$
$SNtP_6(sntdp)$	1.4488220738476987965...	$1.617\cdot10^{-56}$
$SNtP_6(snttp)$	1.4498145515325115306...	$2.381\cdot10^{-56}$
$SNtP_7(sntp,2)$	0.10067792162571842448...	$1.817\cdot10^{-58}$
$SNtP_7(sntdp,2)$	0.10518174020177436646...	$3.441\cdot10^{-58}$
$SNtP_7(snttp,2)$	0.10530572828545769371...	$5.065\cdot10^{-58}$
$SNtP_7(sntp,3)$	0.0028612773389160206688...	$3.786\cdot10^{-60}$
$SNtP_7(sntdp,3)$	0.0034992898623021328169...	$7.170\cdot10^{-60}$
$SNtP_7(snttp,3)$	0.0035130621711970344353...	$1.055\cdot10^{-59}$
$SNtP_8(sntp,1)$	4.1541824026937241559...	$1.768\cdot10^{-53}$
$SNtP_8(sntdp,1)$	5.7207762058536591328...	$3.348\cdot10^{-53}$
$SNtP_8(snttp,1)$	5.762598971655286527...	$4.928\cdot10^{-53}$
$SNtP_8(sntp,2)$	4.6501436396625714979...	$7.779\cdot10^{-52}$
$SNtP_8(sntdp,2)$	6.4108754573319424904...	$1.473\cdot10^{-51}$
$SNtP_8(snttp,2)$	6.6209627683988696642...	$2.168\cdot10^{-51}$
$SNtP_8(sntp,3)$	4.126169449994612004...	$3.345\cdot10^{-50}$
$SNtP_8(sntdp,3)$	7.9082950789773488585...	$6.334\cdot10^{-50}$
$SNtP_8(snttp,3)$	8.7576630556464774578...	$9.324\cdot10^{-50}$
$SNtP_9(sntp)$	1.4214338438480314719...	$3.866\cdot10^{-61}$
$SNtP_9(sntdp)$	1.0466424860358234656...	$1.040\cdot10^{-152}$
$SNtP_9(snttp)$	1.055311342837214453...	$5.902\cdot10^{-223}$
$SNtP_{10}(sntp,1)$	2.218747505497683865...	$4.184\cdot10^{-32}$
$SNtP_{10}(sntdp,1)$	1.2778419356685079191...	$8.745\cdot10^{-71}$
$SNtP_{10}(snttp,1)$	1.2414085582609377035...	$8.294\cdot10^{-114}$
$SNtP_{10}(sntp,2)$	1.5096212187703265236...	$8.902\cdot10^{-34}$
$SNtP_{10}(sntdp,2)$	0.59144856965778964572...	$9.826\cdot10^{-73}$
$SNtP_{10}(snttp,2)$	0.58415307582503530822...	$6.331\cdot10^{-116}$
$SNtP_{10}(sntp,3)$	1.2260357559936064516...	$1.894\cdot10^{-35}$
$SNtP_{10}(sntdp,3)$	0.28337095760325627718...	$1.104\cdot10^{-74}$

Continued on next page

Name	Constant value	Value the last term
$SNtP_{10}(snttp,3)$	0.28191104877850708976...	$4.833\cdot10^{-118}$
$SNtP_{11}(sntp,1)$	1.3610247806676889023...	$6.039\cdot10^{-33}$
$SNtP_{11}(sntdp,1)$	0.71307955647334199639...	$9.218\cdot10^{-72}$
$SNtP_{11}(snttp,1)$	0.69819605640188698911...	$7.219\cdot10^{-115}$
$SNtP_{11}(sntp,2)$	0.98733649403689727251...	$1.285\cdot10^{-34}$
$SNtP_{11}(sntdp,2)$	0.33523656456978691915...	$1.036\cdot10^{-73}$
$SNtP_{11}(snttp,2)$	0.33225730607103958582...	$5.510\cdot10^{-117}$
$SNtP_{11}(sntp,3)$	0.83342721544339356637...	$2.734\cdot10^{-36}$
$SNtP_{11}(sntdp,3)$	0.16190395280891818924...	$1.164\cdot10^{-75}$
$SNtP_{11}(snttp,3)$	0.16130786627737647051...	$4.206\cdot10^{-119}$

6.8 Smarandache–Cira constants

The authors did not prove the convergence towards each constant. We let it as possible research for the interested readers. With the program SC 2.70 calculate vectors top 113 terms numbers containing Smarandache–Cira sequences of order two and three, $n := 1..113$, $sc2 := SC(n,2)$ and $sc3 := SC(n,3)$ given on page 81. We note with $m := last(sc2) = last(sc3)$.

$$sc2^{\mathrm{T}} \to (1\ 2\ 3\ 2\ 5\ 3\ 7\ 4\ 3\ 5\ 11\ 3\ 13\ 7\ 5\ 4\ 17\ 3\ 19\ 5\ 7\ 11\ 23\ 4\ 5\ 13\ 6\ 7\ 29\ 5\ 31\ 4\ 11\ 17\ 7\ 3\ 37\ 19\ 13\ 5\ 41\ 7\ 43\ 11\ 5\ 23\ 47\ 4\ 7\ 5\ 17\ 13\ 53\ 6\ 11\ 7\ 19\ 29\ 59\ 5\ 61\ 31\ 7\ 4\ 13\ 11\ 67\ 17\ 23\ 7\ 71\ 4\ 73\ 37\ 5\ 19\ 11\ 13\ 79\ 5\ 6\ 41\ 83\ 7\ 17\ 43\ 29\ 11\ 89\ 5\ 13\ 23\ 31\ 47\ 19\ 4\ 97\ 7\ 11\ 5\ 101\ 17\ 103\ 13\ 7\ 53\ 107\ 6\ 109\ 11\ 37\ 7\ 113)$$

$$sc3^{\mathrm{T}} \to (1\ 2\ 3\ 2\ 5\ 3\ 7\ 2\ 3\ 5\ 11\ 3\ 13\ 7\ 5\ 4\ 17\ 3\ 19\ 5\ 7\ 11\ 23\ 3\ 5\ 13\ 3\ 7\ 29\ 5\ 31\ 4\ 11\ 17\ 7\ 3\ 37\ 19\ 13\ 5\ 41\ 7\ 43\ 11\ 5\ 23\ 47\ 4\ 7\ 5\ 17\ 13\ 53\ 3\ 11\ 7\ 19\ 29\ 59\ 5\ 61\ 31\ 7\ 4\ 13\ 11\ 67\ 17\ 23\ 7\ 71\ 3\ 73\ 37\ 5\ 19\ 11\ 13\ 79\ 5\ 6\ 41\ 83\ 7\ 17\ 43\ 29\ 11\ 89\ 5\ 13\ 23\ 31\ 47\ 19\ 4\ 97\ 7\ 11\ 5\ 101\ 17\ 103\ 13\ 7\ 53\ 107\ 3\ 109\ 11\ 37\ 7\ 113)$$

$$SC_1(sc2) = \sum_{k=1}^{m} \frac{1}{sc2_k!} float, 20 \to 3.4583335851391576045\ ,$$

$$SC_1(sc3) = \sum_{k=1}^{m} \frac{1}{sc3_k!} float, 20 \to 4.6625002518058242712\ ,$$

$$SC_2(sc2) = \sum_{k=1}^{m} \frac{sc2_k}{k!} float, 20 \to 2.6306646909747437367\ ,$$

$$SC_2(sc3) = \sum_{k=1}^{m} \frac{sc3_k}{k!} float, 20 \to 2.6306150878001405621\ ,$$

$$SC_3(sc2) = \sum_{k=1}^{m} \frac{1}{\prod_{j=1}^{k} sc2_k} float, 20 \to 1.7732952904854629675,$$

$$SC_31(sc3) = \sum_{k=1}^{m} \frac{1}{\prod_{j=1}^{k} sc3_k} float, 20 \to 1.7735747079550529244,$$

$$SC_4(sc2, 1) = \sum_{k=1}^{m} \frac{k}{\prod_{j=1}^{k} sc2_k} float, 20 \to 2.9578888874303295232,$$

$$SC_4(sc3, 1) = \sum_{k=1}^{m} \frac{k}{\prod_{j=1}^{k} sc3_k} float, 20 \to 2.9602222193051036183,$$

$$SC_4(sc2, 2) = \sum_{k=1}^{m} \frac{k^2}{\prod_{j=1}^{k} sc2_k} float, 20 \to 6.5084767524852643905,$$

$$SC_4(sc3, 2) = \sum_{k=1}^{m} \frac{k^2}{\prod_{j=1}^{k} sc3_k} float, 20 \to 6.5280646160816429818,$$

$$SC_4(sc2, 3) = \sum_{k=1}^{m} \frac{k^3}{\prod_{j=1}^{k} sc2_k} float, 20 \to 18.554294952927603195,$$

$$SC_4(sc3, 3) = \sum_{k=1}^{m} \frac{k^3}{\prod_{j=1}^{k} sc3_k} float, 20 \to 18.719701016966392423,$$

$$SC_5(sc2) = \sum_{k=1}^{m} \frac{(-1)^{k+1} sc2_k}{k!} float, 20 \to 0.45546350985738070916\ ,$$

$$SC_5(sc3) = \sum_{k=1}^{m} \frac{(-1)^{k+1} sc3_k}{k!} float, 20 \to 0.45551311303198388376\ ,$$

$$SC_6(sc2) = \sum_{k=1}^{m} \frac{sc2_k}{(k+1)!} float, 20 \to 0.98272529215953150861\ ,$$

$$SC_6(sc3) = \sum_{k=1}^{m} \frac{sc3_k}{(k+1)!} float, 20 \to 0.98271978069568671144\ ,$$

$$SC_7(sc2,2) = \sum_{k=2}^{m} \frac{sc2_k}{(k+2)!} float, 20 \to 0.11219805918720652342\ ,$$

$$SC_7(sc2,3) = \sum_{k=2}^{m} \frac{sc3_k}{(k+2)!} float, 20 \to 0.1121975080408220437\ ,$$

$$SC_7(sc2,3) = \sum_{k=3}^{m} \frac{sc2_k}{(k+3)!} float, 20 \to 0.0046978036116398886027\ ,$$

$$SC_7(sc3,3) = \sum_{k=3}^{m} \frac{sc3_k}{(k+3)!} float, 20 \to 0.0046977535074231177193\ ,$$

$$SC_8(sc2,1) = \sum_{k=1}^{m} \frac{sc2_k}{(k-1)!} float, 20 \to 5.0772738578906499358\ ,$$

$$SC_8(sc3,1) = \sum_{k=1}^{m} \frac{sc3_k}{(k-1)!} float, 20 \to 5.076877032493824539\ ,$$

$$SC_8(sc2,2) = \sum_{k=2}^{m} \frac{sc2_k}{(k-2)!} float, 20 \to 7.0229729491778632583\ ,$$

$$SC_8(sc3,2) = \sum_{k=2}^{m} \frac{sc3_k}{(k-2)!} float, 20 \to 7.0201951714000854806\ ,$$

$$SC_8(sc2,3) = \sum_{k=3}^{m} \frac{sc2_k}{(k-3)!} float, 20 \to 8.3304435839100330540\ ,$$

$$SC_8(sc3,3)=\sum_{k=3}^{m}\frac{sc3_k}{(k-3)!}float,20\to 8.3137769172433663874\ ,$$

$$SC_9(sc2)=\sum_{k=1}^{m}\frac{1}{\sum\limits_{j=1}^{k} sc2_k!}float,20\to 1.5510516488142853476,$$

$$SC_9(sc3)=\sum_{k=1}^{m}\frac{1}{\sum\limits_{j=1}^{k} sc3_k!}float,20\to 1.5510540589248305324,$$

$$SC_{10}(sc2,1)=\sum_{k=1}^{m}\frac{1}{sc2_k\sqrt{sc2_k!}}float,20\to 3.2406242284919649811,$$

$$SC_{10}(sc3,1)=\sum_{k=1}^{m}\frac{1}{sc3_k\sqrt{sc3_k!}}float,20\to 4.1028644277885635587,$$

$$SC_{10}(sc2,2)=\sum_{k=1}^{m}\frac{1}{sc2_k^2\sqrt{sc2_k!}}float,20\to 1.7870808548071955299,$$

$$SC_{10}(sc3,2)=\sum_{k=1}^{m}\frac{1}{sc3_k^2\sqrt{sc3_k!}}float,20\to 2.1492832287173527766,$$

$$SC_{10}(sc2,3)=\sum_{k=1}^{m}\frac{1}{sc2_k^3\sqrt{sc2_k!}}float,20\to 1.3045045248690711166,$$

$$SC_{10}(sc3,3)=\sum_{k=1}^{m}\frac{1}{sc3_k^3\sqrt{sc3_k!}}float,20\to 1.4584084801526035601,$$

$$SC_{11}(sc2,1)=\sum_{k=1}^{m}\frac{1}{sc2_k\sqrt{(sc2_k+1)!}}float,20\to 1.8299218542898376035,$$

$$SC_{11}(sc3,1)=\sum_{k=1}^{m}\frac{1}{sc3_k\sqrt{(sc3_k+1)!}}float,20\to 2.2987446364307715563,$$

$$SC_{11}(sc2,2)=\sum_{k=1}^{m}\frac{1}{sc2_k^2\sqrt{(sc2_k+1)!}}\mathit{float},20\rightarrow 1.1168191325442037858,$$

$$SC_{11}(sc3,2)=\sum_{k=1}^{m}\frac{1}{sc3_k^2\sqrt{(sc3_k+1)!}}\mathit{float},20\rightarrow 1.3139933527909721081,$$

$$SC_{11}(sc2,3)=\sum_{k=1}^{m}\frac{1}{sc2_k^3\sqrt{(sc2_k+1)!}}\mathit{float},20\rightarrow 0.87057913102041407346,$$

$$SC_{11}(sc3,3)=\sum_{k=1}^{m}\frac{1}{sc3_k^3\sqrt{(sc3_k+1)!}}\mathit{float},20\rightarrow 0.95493621492446577773.$$

6.9 Smarandache–X-nacci constants

Let $n := 1..80$ be and the commands $sf_n := SF(n)$, $str_n := STr(n)$ and $ste_n := STe(n)$, then

$sf^{\mathrm{T}} \rightarrow$ (1 3 4 6 5 12 8 6 12 15 10 12 7 24 20 12 9 12 18 30
8 30 24 12 25 21 36 24 14 60 30 24 20 9 40 12 19 18 28 30
20 24 44 30 60 24 16 12 56 75 36 42 27 36 10 24 36 42 58 60
15 30 24 48 35 60 68 18 24 120 70 12 37 57 100 18 40 84 78 60)

$str^{\mathrm{T}} \rightarrow$ (1 3 7 4 14 7 5 7 9 19 8 7 6 12 52 15 28 12 18 31 12 8
29 7 30 39 9 12 77 52 14 15 35 28 21 12 19 28 39 31 35 12 82
8 52 55 29 64 15 52 124 39 33 35 14 12 103 123 64 52 68 60
12 15 52 35 100 28 117 31 132 12 31 19 52 28 37 39 18 31)

$ste^{\mathrm{T}} \rightarrow$ (1 3 6 4 6 9 8 5 9 13 20 9 10 8 6 10 53 9 48 28 18 20 35
18 76 10 9 8 7 68 20 15 20 53 30 9 58 48 78 28 19 18 63 20 68
35 28 18 46 108 76 10 158 9 52 8 87 133 18 68 51 20 46 35 78
20 17 138 35 30 230 20 72 58 76 48 118 78 303 30)

$$SX_1(s)=\sum_{k=1}^{m}\frac{1}{s_k!}\ . \tag{6.76}$$

$$SX_2(s)=\sum_{k=1}^{m}\frac{s_k}{k!}\ . \tag{6.77}$$

$$SX_3(s)=\sum_{k=1}^{m}\frac{1}{\prod_{j=1}^{k}s_j}\ . \tag{6.78}$$

$$SX_4(s,\alpha)=\sum_{k=1}^{m}\frac{k^\alpha}{\prod_{j=1}^{k}s_j}\ ,\ \text{where}\ \ \alpha\in\mathbb{N}^*\ . \tag{6.79}$$

$$SX_5(s)=\sum_{k=1}^{m}\frac{(-1)^{k+1}s_k}{k!} \tag{6.80}$$

$$SX_6(s)=\sum_{k=1}^{m}\frac{s_k}{(k+1)!} \tag{6.81}$$

$$SX_7(s,r)=\sum_{k=r}^{m}\frac{s_k}{(k+r)!}\ ,\ \text{where}\ \ r\in\mathbb{N}^*\ . \tag{6.82}$$

$$SX_8(s,r)=\sum_{k=r}^{m}\frac{s_k}{(k-r)!}\ ,\ \text{where}\ \ r\in\mathbb{N}^*\ . \tag{6.83}$$

$$SX_9(s)=\sum_{k=1}^{m}\frac{1}{\sum_{j=1}^{k}s_k!} \tag{6.84}$$

$$SX_{10}(s,\alpha)=\sum_{k=1}^{m}\frac{1}{s_k^\alpha\sqrt{s_k!}}\ ,\ \text{where}\ \ \alpha\in\mathbb{N}^*\ . \tag{6.85}$$

$$SX_{11}(s,\alpha)=\sum_{k=1}^{m}\frac{1}{s_k^\alpha\sqrt{(s_k+1)!}}\ ,\ \text{where}\ \ \alpha\in\mathbb{N}^*\ . \tag{6.86}$$

In the formulas (6.76–6.86) will replace s with *sf* or *str* or *ste* and $m := last(sf) = last(str) = last(ste)$.

The authors did not prove the convergence towards each constant. We let it as possible research for the interested readers.

Table 6.6: Smarandache–X-nacci constants

Name	Constant value	Value the last term
$SX_1(sf)$	1.2196985417298194908...	$1.202\cdot10^{-82}$

Continued on next page

Name	Constant value	Value the last term
$SX_1(str)$	1.2191275540999213351...	$1.216 \cdot 10^{-34}$
$SX_1(ste)$	1.2211513449585368892...	$3.770 \cdot 10^{-33}$
$SX_2(sf)$	3.4767735904805818975...	$8.383 \cdot 10^{-118}$
$SX_2(str)$	3.9609181504757024515...	$4.331 \cdot 10^{-118}$
$SX_2(ste)$	3.7309068817634133077...	$4.192 \cdot 10^{-118}$
$SX_3(sf)$	1.4335990041360201401...	$1.513 \cdot 10^{-108}$
$SX_3(str)$	1.3938571352678434235...	$7.402 \cdot 10^{-108}$
$SX_3(ste)$	1.4053891469804807777...	$5.461 \cdot 10^{-109}$
$SX_4(sf,1)$	1.9877450439442829197...	$1.210 \cdot 10^{-106}$
$SX_4(str,1)$	1.8623249618930151417...	$5.922 \cdot 10^{-106}$
$SX_4(ste,1)$	1.9022896785778318923...	$4.369 \cdot 10^{-107}$
$SX_4(sf,2)$	3.3850953926486127438...	$9.681 \cdot 10^{-105}$
$SX_4(str,2)$	2.9794588765640621423...	$4.737 \cdot 10^{-104}$
$SX_4(ste,2)$	3.1247358165606852605...	$3.495 \cdot 10^{-105}$
$SX_4(sf,3)$	7.2154954684533914439...	7.74510^{-103}
$SX_4(str,3)$	5.8572332489153350088...	$3.790 \cdot 10^{-102}$
$SX_4(ste,3)$	6.4153627205469027224...	$2.796 \cdot 10^{-103}$
$SX_5(sf)$	–0.0568656797521010866683...	$-8.383 \cdot 10^{-118}$
$SX_5(str)$	0.6077826491902020422...	$-4.331 \cdot 10^{-118}$
$SX_5(ste)$	0.3723183298914126735...	$-4.192 \cdot 10^{-118}$
$SX_6(sf)$	1.2262107161454250477...	$1.035 \cdot 10^{-119}$
$SX_6(str)$	1.3459796054481592511...	$5.347 \cdot 10^{-120}$
$SX_6(ste)$	1.2936674214665211769...	$5.175 \cdot 10^{-120}$
$SX_7(sf,2)$	0.1679803821914295440 9...	$1.262 \cdot 10^{-121}$
$SX_7(str,2)$	0.1918562519548785314 6...	$6.521 \cdot 10^{-122}$
$SX_7(ste,2)$	0.1819929256849398436 4...	$6.311 \cdot 10^{-122}$
$SX_7(sf,3)$	0.0069054909490509753782...	$1.52 \cdot 10^{-123}$
$SX_7(str,3)$	0.0108839605300192862 62...	$7.857 \cdot 10^{-124}$
$SX_7(ste,3)$	0.0093029461977309121111...	$7.604 \cdot 10^{-124}$
$SX_8(sf,1)$	7.3209769507255575585...	$6.707 \cdot 10^{-116}$
$SX_8(str,1)$	8.8169446348716671749...	$3.465 \cdot 10^{-116}$
$SX_8(ste,1)$	8.0040346392438852968...	$3.353 \cdot 10^{-116}$
$SX_8(sf,2)$	11.411117402547284927...	$5.298 \cdot 10^{-114}$
$SX_8(str,2)$	14.678669992110622025...	$2.737 \cdot 10^{-114}$
$SX_8(ste,2)$	12.450777109170763294...	$2.649 \cdot 10^{-114}$
$SX_8(sf,3)$	14.903259849189180761...	$4.133 \cdot 10^{-112}$
$SX_8(str,3)$	19.449822942788797955...	$2.135 \cdot 10^{-112}$
$SX_8(ste,3)$	14.890603169310654296...	$2.066 \cdot 10^{-112}$

Continued on next page

Name	Constant value	Value the last term
$SX_9(sf)$	1.1775948782312684824...	$2.103 \cdot 10^{-123}$
$SX_9(str)$	1.1432524801852870116...	$1.340 \cdot 10^{-95}$
$SX_9(ste)$	1.1462530152136221219...	$4.886 \cdot 10^{-88}$
$SX_{10}(sf,1)$	1.2215596514691068605...	$1.827 \cdot 10^{-43}$
$SX_{10}(str,1)$	1.2239155500269214276...	$3.557 \cdot 10^{-19}$
$SX_{10}(ste,1)$	1.2300096122076512655...	$2.047 \cdot 10^{-18}$
$SX_{10}(sf,2)$	1.0643380628436674107...	$3.045 \cdot 10^{-45}$
$SX_{10}(str,2)$	1.0645200726839585165...	$1.148 \cdot 10^{-20}$
$SX_{10}(ste,2)$	1.0656390763277979581...	$6.822 \cdot 10^{-20}$
$SX_{10}(sf,3)$	1.0194514801361011603...	$5.075 \cdot 10^{-47}$
$SX_{10}(str,3)$	1.0194518909553334484...	$3.702 \cdot 10^{-22}$
$SX_{10}(ste,3)$	1.0196556717297023297...	$2.274 \cdot 10^{-21}$
$SX_{11}(sf,1)$	0.81140316439268935525...	$2.340 \cdot 10^{-44}$
$SX_{11}(str,1)$	0.81207729582854199505...	$6.289 \cdot 10^{-20}$
$SX_{11}(ste,1)$	0.81447979678562282739...	$3.676 \cdot 10^{-19}$
$SX_{11}(sf,2)$	0.73793638461692431348...	$3.899 \cdot 10^{-46}$
$SX_{11}(str,2)$	0.7379744330358397038...	$2.029 \cdot 10^{-21}$
$SX_{11}(ste,2)$	0.73841432833681481939...	$6.822 \cdot 10^{-20}$
$SX_{11}(sf,3)$	0.71654469002152894246...	$6.498 \cdot 10^{-48}$
$SX_{11}(str,3)$	0.71654048115486167686...	$6.544 \cdot 10^{-23}$
$SX_{11}(ste,3)$	0.716620239259230028...	$2.274 \cdot 10^{-21}$

6.10 The Family of Metallic Means

The family of *Metallic Means* (whom most prominent members are the *Golden Mean*, *Silver Mean*, *Bronze Mean*, *Nickel Mean*, *Copper Mean*, etc.) comprises every quadratic irrational number that is the positive solution of one of the algebraic equations

$$x^2 - n \cdot x - 1 = 0 \text{ or } x^2 - x - n = 0 ,$$

where $n \in \mathbb{N}$. All of them are closely related to quasi-periodic dynamics, being therefore important basis of musical and architectural proportions. Through the analysis of their common mathematical properties, it becomes evident that they interconnect different human fields of knowledge, in the sense defined in "*Paradoxist Mathematics*". Being irrational numbers, in applications to different scientific disciplines, they have to be approximated by ratios of integers – which is the goal of this paper, [de Spinadel, 1998].

The solutions of equation $n^2 - n \cdot x - 1 = 0$ are:

$$x^2 - n \cdot x - 1 \; solve, x \rightarrow \begin{pmatrix} \frac{n + \sqrt{n^2+4}}{2} \\ \frac{n - \sqrt{n^2+4}}{2} \end{pmatrix}. \tag{6.87}$$

If we denote by $s_1(n)$ positive solution, then for $n := 1..10$ we have the solutions:

$$s_1(n) \rightarrow \begin{pmatrix} \frac{\sqrt{5}+2}{2} \\ \sqrt{2}+1 \\ \frac{\sqrt{13}+3}{2} \\ \sqrt{5}+2 \\ \frac{\sqrt{29}+5}{2} \\ \sqrt{10}+3 \\ \frac{\sqrt{53}+7}{2} \\ \sqrt{17}+4 \\ \frac{\sqrt{85}+9}{2} \\ \sqrt{26}+5 \end{pmatrix} = \begin{pmatrix} 1.618033988749895 \\ 2.414213562373095 \\ 3.302775637731995 \\ 4.236067977499790 \\ 5.192582403567252 \\ 6.162277660168380 \\ 7.140054944640259 \\ 8.123105625617661 \\ 9.109772228646444 \\ 10.099019513592784 \end{pmatrix}.$$

The solutions of equation $n^2 - x - n = 0$ are:

$$x^2 - x - n \; solve, x \rightarrow \begin{pmatrix} \frac{1 + \sqrt{4n+1}}{2} \\ \frac{1 - \sqrt{4n+1}}{2} \end{pmatrix}. \tag{6.88}$$

If we denote by $s_2(n)$ positive solution, then for $n := 1..10$ we have the solutions:

$$s_2(n) \rightarrow \begin{pmatrix} \frac{\sqrt{5}+2}{2} \\ 2 \\ \frac{\sqrt{13}+1}{2} \\ \frac{\sqrt{17}+1}{2} \\ \frac{\sqrt{21}+1}{2} \\ 3 \\ \frac{\sqrt{29}+1}{2} \\ \frac{\sqrt{33}+1}{2} \\ \frac{\sqrt{37}+1}{2} \\ \frac{\sqrt{41}+5}{2} \end{pmatrix} = \begin{pmatrix} 1.6180339887498950 \\ 2.0000000000000000 \\ 2.3027756377319950 \\ 2.5615528128088303 \\ 2.7912878474779200 \\ 3.0000000000000000 \\ 3.1925824035672520 \\ 3.3722813232690143 \\ 3.5413812651491097 \\ 3.7015621187164243 \end{pmatrix}.$$

Chapter 7

Numerical Carpet

7.1 Generating Cellular Matrices

Function 7.1. Concatenation function of two numbers in the base on numeration 10.

$$conc(n,m) := \left|\begin{array}{l} return\ n\cdot 10\ \ if\ \ m{=}0 \\ return\ n\cdot 10^{nrd(m,10)} + m\ otherwise \end{array}\right.$$

Examples of calling the function *conc*: $conc(123,78) \to 12378$, $conc(2,3) \to 23$, $conc(2,35) \to 235$, $conc(23,5) \to 235$, $conc(0,12) \to 12$, $conc(13,0) \to 130$.

Program 7.2. Concatenation program in base 10 of all the elements on a line, all the matrix lines. The result is a vector. The origin of indexes is 1, i.e.

$$ORIGIN := 1$$

```
concM(M) := | c ← cols(M)
            | for k ∈ 1..rows(M)
            |   | v_k ← M_{k,1}
            |   | for j ∈ 2..c−1
            |   |   | v_k ← conc(v_k, M_{k,j})  if M_{k,j} ≠ 0
            |   |   | otherwise
            |   |   |   | sw ← 0
            |   |   |   | for i ∈ j+1..c
            |   |   |   |   if M_{k,i} ≠ 0
            |   |   |   |     | sw ← 1
            |   |   |   |     | break
            |   |   |   | v_k ← conc(v_k, M_{k,j})  if sw=1
            |   | v_k ← conc(v_k, M_{k,c})  if M_{k,c} ≠ 0
            | return v
```

Matrix concatenation program does not concatenate on zero if on that line after the zero we only have zeros on every column. Obviously, zeros preceding a non–zero number have no value.

Examples of calling the program *concM*:

$$M := \begin{pmatrix} 1 & 3 & 5 \\ & 11 & 13 \\ 17 & & 23 \end{pmatrix} \quad concM(M) = \begin{pmatrix} 135 \\ 1113 \\ 17023 \end{pmatrix},$$

$$M := \begin{pmatrix} & 1 & \\ 1 & 2 & 1 \\ & 1 & \end{pmatrix} \quad concM(M) = \begin{pmatrix} 1 \\ 121 \\ 1 \end{pmatrix}.$$

Using the function *conc* and the matrix concatenation program one can generate carpet numbers. For generating carpet numbers we present a program which generates *cellular matrices*. Let a vector v of m size smaller than the size of the cell matrix that is generated.

Definition 7.3. By *cellular matrix* we understand a square matrix having an odd number of lines. Matrix values are only the values of the vector v and eventually 0. The display of vector values v is made by a rule that is based on a function f.

Program 7.4. Program for generating cellular matrices, of $n \times n$ (n odd) size, with the values of the vector v following the rule imposed by the function f.

$$\begin{array}{l|l} GMC(v,n,f) := & return\ \text{"Nr. cols and rows odd"}\ \ if\ \ \mod(n,2)=0 \\ & m \leftarrow \frac{n+1}{2} \\ & A_{n,n} \leftarrow 0 \\ & for\ k \in 1..n \\ & \quad for\ j \in 1..n \\ & \qquad \left| \begin{array}{l} q \leftarrow |k-m| \\ s \leftarrow |j-m| \\ A_{k,j} \leftarrow v_{f(q,s)+1}\ \ if\ f(q,s)+1 \le last(v) \end{array} \right. \\ & return\ A \end{array}$$

Example 7.5. for calling the program to generate cellular matrices. Let the vec-

tor $v = (13\ \ 7\ \ 1)^{\mathrm{T}}$ and function $f_1(q,s) := q + s$. Thus we have:

$$M := GMC(v,7,f_1) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 7 & 1 & 0 & 0 \\ 0 & 1 & 7 & 13 & 7 & 1 & 0 \\ 0 & 0 & 1 & 7 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$N := concM(M) \to \begin{pmatrix} 0 \\ 1 \\ 171 \\ 171371 \\ 171 \\ 1 \\ 0 \end{pmatrix}.$$

With the command sequence: $k := 1..last(N)$,

$$IsPrime(N_k) \to \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},\ factorN_k \to \begin{pmatrix} 0 \\ 1 \\ 3^2 \cdot 19 \\ 409 \cdot 419 \\ 3^2 \cdot 19 \\ 1 \\ 0 \end{pmatrix},$$

we can study the nature of these numbers by using functions Mathcad *factor*, *IsPrime*, etc.

As it turns, the function $f_1(q,s) = q + s$ will generate the matrix

$$M := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & v_3 & 0 & 0 & 0 \\ 0 & 0 & v_3 & v_2 & v_3 & 0 & 0 \\ 0 & v_3 & v_2 & v_1 & v_2 & v_3 & 0 \\ 0 & 0 & v_3 & v_2 & v_3 & 0 & 0 \\ 0 & 0 & 0 & v_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

therefore if the vector has m ($m \le n$) elements and the generated matrix size is $2n+1$ it will result a matrix that has in its center v_1, then, around v_2 and so on.

By concatenating the matrix, the following carpet number will result:

$$concM(M) = \begin{pmatrix} 0 \\ \overline{v_3} \\ \overline{v_3 v_2 v_3} \\ \overline{v_3 v_2 v_1 v_2 v_3} \\ \overline{v_3 v_2 v_3} \\ v_3 \\ 0 \end{pmatrix}$$

or, if we concatenate the matrix $M_1 := submatrix(M, 1, rows(M), 1, n+1)$ will result:

$$concM(M_1) = \begin{pmatrix} 0 \\ \overline{v_3} \\ \overline{v_3 v_2} \\ \overline{v_3 v_2 v_1} \\ \overline{v_3 v_2} \\ v_3 \\ 0 \end{pmatrix},$$

or, if we concatenate the matrix $M_2 := submatrix(M, 1, n+1, 1, n+1)$ will result:

$$concM(M_2) = \begin{pmatrix} 0 \\ \overline{v_3} \\ \overline{v_3 v_2} \\ v_3 v_2 v_1 \end{pmatrix}.$$

7.2 Carpet Numbers Study

As seen, the function *conc*, the program *GCM* with the function f and the subprogram *concM* allow us to generate a great diversity of carpet numbers. We offer a list of functions f for generating *cellular matrices* with interesting

structures.:

$$
\begin{aligned}
f_1(k,j) &= k+j\ , && (7.1)\\
f_2(k,j) &= \left|k-j\right|\ , && (7.2)\\
f_3(k,j) &= \max(k,j)\ , && (7.3)\\
f_4(k,j) &= \min(k,j)\ , && (7.4)\\
f_5(k,j) &= k\cdot j\ , && (7.5)\\
f_6(k,j) &= \left\lfloor\frac{k+1}{j+1}\right\rfloor\ , && (7.6)\\
f_7(k,j) &= \left\lceil\frac{k}{j+1}\right\rceil\ , && (7.7)\\
f_8(k,j) &= \min(\left|k-j\right|,k,j)\ , && (7.8)\\
f_9(k,j) &= \left\lceil\frac{\max(k,j)+1}{\min(k,j)+1}\right\rceil\ , && (7.9)\\
f_{10}(k,j) &= \min(S(k+1),S(j+1)\ , && (7.10)\\
f_{11}(k,j) &= \left\lceil k\cdot\sin(j)^3+j\cos(k)^3\right\rceil\ , && (7.11)\\
f_{12}(k,j) &= \left\lfloor\frac{k+2j}{3}\right\rfloor\ . && (7.12)
\end{aligned}
$$

where S is the Smarandache function.

Let the vector $v:=(1\ \ 3\ \ 9\ \ 7)^{\mathrm{T}}$, therefore the vector size is $m=4$, cell matrices that we generate to be of size 9.

1. The case for generating function f_1 given by formula (7.1).

$$
M_1:=GMC(v,9,f_1)=\begin{pmatrix}
0&0&0&0&0&0&0&0&0\\
0&0&0&0&7&0&0&0&0\\
0&0&0&7&9&7&0&0&0\\
0&0&7&9&3&9&7&0&0\\
0&7&9&3&1&3&9&7&0\\
0&0&7&9&3&9&7&0&0\\
0&0&0&7&9&7&0&0&0\\
0&0&0&0&7&0&0&0&0\\
0&0&0&0&0&0&0&0&0
\end{pmatrix}
$$

$$N_1 := concM(M_1) \rightarrow \begin{pmatrix} 0 \\ 7 \\ 797 \\ 79397 \\ 7931397 \\ 79397 \\ 797 \\ 7 \\ 0 \end{pmatrix}$$

$$k := 1..last(N_1)\ ,\quad {N_1}_k factor \rightarrow \begin{pmatrix} 0 \\ 7 \\ 797 \\ 79397 \\ 3 \cdot 53 \cdot 83 \cdot 601 \\ 79397 \\ 797 \\ 7 \\ 0 \end{pmatrix}.$$

2. The case for generating function f_2 given by formula (7.2).

$$M_2 := GMC(v, 9, f_2) = \begin{pmatrix} 1 & 3 & 9 & 7 & 0 & 7 & 9 & 3 & 1 \\ 3 & 1 & 3 & 9 & 7 & 9 & 3 & 1 & 3 \\ 9 & 3 & 1 & 3 & 9 & 3 & 1 & 3 & 9 \\ 7 & 9 & 3 & 1 & 3 & 1 & 3 & 9 & 7 \\ 0 & 7 & 9 & 3 & 1 & 3 & 9 & 7 & 0 \\ 7 & 9 & 3 & 1 & 3 & 1 & 3 & 9 & 7 \\ 9 & 3 & 1 & 3 & 9 & 3 & 1 & 3 & 9 \\ 3 & 1 & 3 & 9 & 7 & 9 & 3 & 1 & 3 \\ 1 & 3 & 9 & 7 & 0 & 7 & 9 & 3 & 1 \end{pmatrix}$$

$$N_2 := concM(M_2) \rightarrow \begin{pmatrix} 139707931 \\ 313979313 \\ 931393139 \\ 793131397 \\ 7931397 \\ 793131397 \\ 931393139 \\ 313979313 \\ 139707931 \end{pmatrix}$$

$$k := 1..last(N_2)\ ,\quad N_{2_k} factor \rightarrow \begin{pmatrix} 11^2 \cdot 19 \cdot 67 \cdot 907 \\ 3 \cdot 19 \cdot 2347^2 \\ 601 \cdot 1549739 \\ 793131397 \\ 3 \cdot 53 \cdot 83 \cdot 601 \\ 793131397 \\ 601 \cdot 1549739 \\ 3 \cdot 19 \cdot 2347^2 \\ 11^2 \cdot 19 \cdot 67 \cdot 907 \end{pmatrix}.$$

3. The case for generating function f_3 given by formula (7.3).

$$M_3 := GMC(v, 9, f_3) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 0 \\ 0 & 7 & 9 & 9 & 9 & 9 & 9 & 7 & 0 \\ 0 & 7 & 9 & 3 & 3 & 3 & 9 & 7 & 0 \\ 0 & 7 & 9 & 3 & 1 & 3 & 9 & 7 & 0 \\ 0 & 7 & 9 & 3 & 3 & 3 & 9 & 7 & 0 \\ 0 & 7 & 9 & 9 & 9 & 9 & 9 & 7 & 0 \\ 0 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$N_3 := concM(M_3) \rightarrow \begin{pmatrix} 0 \\ 7777777 \\ 7999997 \\ 7933397 \\ 7931397 \\ 7933397 \\ 7999997 \\ 7777777 \\ 0 \end{pmatrix}$$

$$k := 1..last(N_3)\ ,\quad N_{3_k} factor \rightarrow \begin{pmatrix} 0 \\ 7 \cdot 239 \cdot 4649 \\ 73 \cdot 109589 \\ 7933397 \\ 3 \cdot 53 \cdot 83 \cdot 601 \\ 7933397 \\ 73 \cdot 109589 \\ 7 \cdot 239 \cdot 4649 \\ 0 \end{pmatrix}.$$

4. The case for generating function f_4 given by formula (7.4).

$$M_4 := GMC(v,9,f_4) = \begin{pmatrix} 0 & 7 & 9 & 3 & 1 & 3 & 9 & 7 & 0 \\ 7 & 7 & 9 & 3 & 1 & 3 & 9 & 7 & 7 \\ 9 & 9 & 9 & 3 & 1 & 3 & 9 & 9 & 9 \\ 3 & 3 & 3 & 3 & 1 & 3 & 3 & 3 & 3 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 3 & 3 & 3 & 3 & 1 & 3 & 3 & 3 & 3 \\ 9 & 9 & 9 & 3 & 1 & 3 & 9 & 9 & 9 \\ 7 & 7 & 9 & 3 & 1 & 3 & 9 & 7 & 7 \\ 0 & 7 & 9 & 3 & 1 & 3 & 9 & 7 & 0 \end{pmatrix}$$

$$N_4 := concM(M_4) \to \begin{pmatrix} 7931397 \\ 779313977 \\ 999313999 \\ 333313333 \\ 111111111 \\ 333313333 \\ 999313999 \\ 779313977 \\ 7931397 \end{pmatrix}$$

$$k := 1..last(N_4)\ , \quad {N_4}_k factor \to \begin{pmatrix} 3 \cdot 53 \cdot 83 \cdot 601 \\ 13 \cdot 5657 \cdot 10597 \\ 263 \cdot 761 \cdot 4993 \\ 19 \cdot 31 \cdot 61 \cdot 9277 \\ 3^2 \cdot 37 \cdot 333667 \\ 19 \cdot 31 \cdot 61 \cdot 9277 \\ 263 \cdot 761 \cdot 4993 \\ 13 \cdot 5657 \cdot 10597 \\ 3 \cdot 53 \cdot 83 \cdot 601 \end{pmatrix}.$$

5. The case for generating function f_5 given by formula (7.5).

$$M_5 := GMC(v,9,f_5) = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 7 & 1 & 7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 9 & 1 & 9 & 0 & 0 & 0 \\ 0 & 7 & 9 & 3 & 1 & 3 & 9 & 7 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 7 & 9 & 3 & 1 & 3 & 9 & 7 & 0 \\ 0 & 0 & 0 & 9 & 1 & 9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 7 & 1 & 7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$N_5 := concM(M_5) \rightarrow \begin{pmatrix} 1 \\ 717 \\ 919 \\ 7931397 \\ 111111111 \\ 7931397 \\ 919 \\ 717 \\ 1 \end{pmatrix}$$

$$k := 1..last(N_5)\ ,\ \ {N_5}_k factor \rightarrow \begin{pmatrix} 1 \\ 3\cdot 239 \\ 919 \\ 3\cdot 53\cdot 83\cdot 601 \\ 3^2\cdot 37\cdot 333667 \\ 3\cdot 53\cdot 83\cdot 601 \\ 919 \\ 3\cdot 239 \\ 1 \end{pmatrix}.$$

7.3 Other Carpet Numbers Study

Obviously, square matrices can be introduced using formulas or manually, and then to apply the concatenation program *concM*. Using the vector containing carpet numbers we can proceed to study the numbers the way we did above.

1. Carpet numbers generated by the series given by formula, [Smarandache, 2014, 1995]:

$$C(n,k) = 4n\prod_{j=1}^{k}(4n-4j+1),\ \text{ for }\ 1 \le k \le n$$

and $C(n,0) = 1$ for any $n \in \mathbb{N}$.

Program 7.6. to generate the matrix M.

$$GenM(D) := \left| \begin{array}{l} for\ \ n \in 1..D \\ \quad for\ k \in 1..n \\ \qquad M_{n,n-k+1} \leftarrow C(n-1,k-1) \\ return\ \ M \end{array} \right.$$

If $D := 7$ and we command $M := GenM(D)$, then we have:

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 & 0 & 0 & 0 \\ 40 & 8 & 1 & 0 & 0 & 0 & 0 \\ 504 & 108 & 12 & 1 & 0 & 0 & 0 \\ 9360 & 1872 & 208 & 16 & 1 & 0 & 0 \\ 198900 & 39780 & 4420 & 340 & 20 & 1 & 0 \\ 5012280 & 1002456 & 111384 & 8568 & 504 & 24 & 1 \end{pmatrix}$$

$$N := concM(M) \to \begin{pmatrix} 1 \\ 41 \\ 4081 \\ 504108121 \\ 93601872208161 \\ 1989003978044203402 01 \\ 5012280100245611138485685042 41 \end{pmatrix}$$

$$k := 1..last(N)\ ,$$

$$N_k\ factor \to \begin{pmatrix} 1 \\ 41 \\ 7 \cdot 11 \cdot 53 \\ 11 \cdot 239 \cdot 191749 \\ 3^2 \cdot 17 \cdot 1367 \cdot 447532511 \\ 83 \cdot 2396390334993016147 \\ 31 \cdot 3169 \cdot 5341781 \cdot 9551362335185180 99 \end{pmatrix}.$$

2. Carpet numbers generated by Pascal triangle. Let the matrix by Pascal's triangle values:

Program 7.7. generating the matrix by Pascal's triangle.

$$Pascal(n) := \left| \begin{array}{l} for\ k \in 1..n+1 \\ \quad for\ j \in 1..k \\ \quad\quad M_{k,j} \leftarrow combin(k-1, j-1) \\ return\ M \end{array} \right.$$

The Pascal program generates a matrix containing Pascal's triangle.

$$M := Pascal(10) =
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 3 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 4 & 6 & 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 5 & 10 & 10 & 5 & 1 & 0 & 0 & 0 & 0 & 0\\
1 & 6 & 15 & 20 & 15 & 6 & 1 & 0 & 0 & 0 & 0\\
1 & 7 & 21 & 35 & 35 & 21 & 7 & 1 & 0 & 0 & 0\\
1 & 8 & 28 & 56 & 70 & 56 & 28 & 8 & 1 & 0 & 0\\
1 & 9 & 36 & 84 & 126 & 126 & 84 & 36 & 9 & 1 & 0\\
1 & 10 & 45 & 120 & 210 & 252 & 210 & 120 & 45 & 10 & 1
\end{pmatrix} .$$

Using the program *concM* for concatenating the components of the matrix, it results the carpet numbers:

$$N := concM(M) \to \begin{pmatrix}
1\\
11\\
121\\
1331\\
14641\\
15101051\\
1615201561\\
172135352171\\
18285670562881\\
193684126126843691\\
11045120210252210120451 01
\end{pmatrix} ,$$

whose decomposition in prime factors is:

$$k := 1..last(N) ,$$

$$N_k\ factor \rightarrow \begin{pmatrix} 1 \\ 11 \\ 11^2 \\ 11^3 \\ 11^4 \\ 7\cdot 2157293 \\ 43\cdot 37562827 \\ 29\cdot 5935701799 \\ 18285670562881 \\ 5647\cdot 34298587945253 \\ 13\cdot 197\cdot 4649\cdot 92768668286052709 \end{pmatrix}.$$

3. Carpet numbers generated by primes.

   Program 7.8. for generating matrix by primes.

$$MPrime(n) := \left| \begin{array}{l} for\ k \in 1..n+1 \\ \quad for\ j \in 1..k \\ \quad\quad M_{k,j} \leftarrow prime_j \\ return\ M \end{array} \right.$$

   Before using the program *MPrime* we have to generate the vector of primes by instruction: *prime* := *SEPC*(100), where we call the program 1.1.

$$M := MPrime(10) = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 3 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 3 & 5 & 7 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 3 & 5 & 7 & 11 & 0 & 0 & 0 & 0 & 0 \\ 2 & 3 & 5 & 7 & 11 & 13 & 0 & 0 & 0 & 0 \\ 2 & 3 & 5 & 7 & 11 & 13 & 17 & 0 & 0 & 0 \\ 2 & 3 & 5 & 7 & 11 & 13 & 17 & 19 & 0 & 0 \\ 2 & 3 & 5 & 7 & 11 & 13 & 17 & 19 & 23 & 0 \\ 2 & 3 & 5 & 7 & 11 & 13 & 17 & 19 & 23 & 29 \end{pmatrix}$$

   Using the program *concM* for concatenating the matrix components we

have the carpet numbers:

$$N := concM(M) \rightarrow \begin{pmatrix} 2 \\ 23 \\ 235 \\ 2357 \\ 235711 \\ 23571113 \\ 2357111317 \\ 235711131719 \\ 23571113171923 \\ 2357111317192329 \end{pmatrix},$$

whose decomposition in prime factors is:

$$k := 1..last(N) ,$$

$$N_k \; factor \rightarrow \begin{pmatrix} 2 \\ 23 \\ 5 \cdot 47 \\ 2357 \\ 7 \cdot 151 \cdot 223 \\ 23 \cdot 29 \cdot 35339 \\ 11 \cdot 214282847 \\ 7 \cdot 4363 \cdot 7717859 \\ 61 \cdot 478943 \cdot 806801 \\ 3 \cdot 4243 \cdot 185176472401 \end{pmatrix}.$$

4. Carpet numbers generated by primes starting with 2, as it follows:

$$M := \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 7 & 11 & 13 & 0 & 0 & 0 & 0 & 0 & 0 \\ 17 & 19 & 23 & 29 & 0 & 0 & 0 & 0 & 0 \\ 31 & 37 & 41 & 43 & 47 & 0 & 0 & 0 & 0 \\ 53 & 59 & 61 & 67 & 71 & 73 & 0 & 0 & 0 \\ 79 & 83 & 89 & 97 & 101 & 103 & 107 & 0 & 0 \\ 109 & 113 & 127 & 131 & 137 & 139 & 149 & 151 & 0 \\ 157 & 163 & 167 & 173 & 179 & 181 & 191 & 193 & 197 \end{pmatrix}$$

Using the program *concM* for concatenating the matrix components we

have the carpet numbers:

$$N := concM(M) \to \begin{pmatrix} 2 \\ 35 \\ 71113 \\ 17192329 \\ 3137414347 \\ 535961677173 \\ 79838997101103107 \\ 109113127131137139149151 \\ 157163167173179181191193197 \end{pmatrix},$$

whose decomposition in prime factors is:

$$k := 1..last(N)\ ,$$

$$N_k\ factor \to \begin{pmatrix} 2 \\ 5\cdot 7 \\ 7\cdot 10159 \\ 7\cdot 11\cdot 223277 \\ 2903\cdot 1080749 \\ 3\cdot 13\cdot 13742607107 \\ 7\cdot 41\cdot 3449\cdot 80656613189 \\ 3\cdot 857\cdot 35039761\cdot 1211194223021 \\ 10491377789\cdot 14980221886391473 \end{pmatrix}.$$

5. Carpet numbers generated by primes starting with 3, as it follows:

$$M := \begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 5 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 11 & 13 & 17 & 0 & 0 & 0 & 0 & 0 & 0 \\ 19 & 23 & 29 & 31 & 0 & 0 & 0 & 0 & 0 \\ 37 & 41 & 43 & 47 & 53 & 0 & 0 & 0 & 0 \\ 59 & 61 & 67 & 71 & 73 & 79 & 0 & 0 & 0 \\ 83 & 89 & 97 & 101 & 103 & 107 & 109 & 0 & 0 \\ 113 & 127 & 131 & 137 & 139 & 149 & 151 & 157 & 0 \\ 163 & 167 & 173 & 179 & 181 & 191 & 193 & 197 & 199 \end{pmatrix}$$

Using the program *concM* for concatenating the matrix components we

have the carpet numbers:

$$N := concM(M) \rightarrow \begin{pmatrix} 3 \\ 57 \\ 111317 \\ 19232931 \\ 3741434753 \\ 596167717379 \\ 838997101103107109 \\ 113127131137139149151157 \\ 163167173179181191193197199 \end{pmatrix},$$

whose decomposition in prime factors is:

$$k := 1..last(N)\ ,$$

$$N_k\ factor \rightarrow \begin{pmatrix} 3 \\ 3\cdot 19 \\ 111317 \\ 3\cdot 6410977 \\ 7\cdot 577\cdot 926327 \\ 13\cdot 45859055183 \\ 3251\cdot 258073546940359 \\ 3\cdot 41\cdot 467\cdot 1969449193731640277 \\ 7\cdot 1931\cdot 47123\cdot 2095837\cdot 122225561597 \end{pmatrix}.$$

7.4 Ulam Matrix

In the Ulam matrices, [Ulam, 1930, Jech, 2003], the natural numbers are placed on a spiral that starts from the center of the matrix. Primes to 169 are in red text. On the main diagonal of the matrix, there are the perfect squares, in blue text

Program 7.9. for generating Ulam matrix.

$$MUlam(n) := \left| \begin{array}{l} return\ "Error.\ n\ \ if\ \ \bmod(n,2)=0 \vee n \le 1 \\ A_{n,n} \leftarrow 0 \\ m \leftarrow \dfrac{n+1}{2} \\ I \leftarrow (m\ \ m) \\ k_f \leftarrow rows(I) \\ for\ s \in 1..n-m \\ \quad \left| \ k_i \leftarrow k_f \right. \end{array} \right.$$

$$
\begin{aligned}
&c \leftarrow s\\
&for\ r \in s-1..-s\\
&\quad I \leftarrow stack[I,(m+r\ \ m+c)]\\
&for\ c \in s-1..-s\\
&\quad I \leftarrow stack[I,(m+r\ \ m+c)]\\
&for\ r \in -s+1..s\\
&\quad I \leftarrow stack[I,(m+r\ \ m+c)]\\
&for\ c \in -s+1..s\\
&\quad I \leftarrow stack[I,(m+r\ \ m+c)]\\
&k_f \leftarrow rows(I)\\
&for\ k \in k_i..k_f\\
&\quad A_{(I_{k,1},I_{k,2})} \leftarrow k\\
&return\ \ A
\end{aligned}
$$

For exemplification, we generate the Ulam matrix of 13 lines and 13 columns by using command $MUlam(13)$.

$$
U := MUlam(13) = \begin{bmatrix}
145 & 144 & 143 & 142 & 141 & 140 & 139 & 138 & 137 & 136 & 135 & 134 & 133\\
146 & 101 & 100 & 99 & 98 & 97 & 96 & 95 & 94 & 93 & 92 & 91 & 132\\
147 & 102 & 65 & 64 & 63 & 62 & 61 & 60 & 59 & 58 & 57 & 90 & 131\\
148 & 103 & 66 & 37 & 36 & 35 & 34 & 33 & 32 & 31 & 56 & 89 & 130\\
149 & 104 & 67 & 38 & 17 & 16 & 15 & 14 & 13 & 30 & 55 & 88 & 129\\
150 & 105 & 68 & 39 & 18 & 5 & 4 & 3 & 12 & 29 & 54 & 87 & 128\\
151 & 106 & 69 & 40 & 19 & 6 & 1 & 2 & 11 & 28 & 53 & 86 & 127\\
152 & 107 & 70 & 41 & 20 & 7 & 8 & 9 & 10 & 27 & 52 & 85 & 126\\
153 & 108 & 71 & 42 & 21 & 22 & 23 & 24 & 25 & 26 & 51 & 84 & 125\\
154 & 109 & 72 & 43 & 44 & 45 & 46 & 47 & 48 & 49 & 50 & 83 & 124\\
155 & 110 & 73 & 74 & 75 & 76 & 77 & 78 & 79 & 80 & 81 & 82 & 123\\
156 & 111 & 112 & 113 & 114 & 115 & 116 & 117 & 118 & 119 & 120 & 121 & 122\\
157 & 158 & 159 & 160 & 161 & 162 & 163 & 164 & 165 & 166 & 167 & 168 & 169
\end{bmatrix}
$$

Using the command $concM(submatrix(U,1,7,1,13) \rightarrow$, for concatenating

the components of submatrix U, we get the carpet Ulam numbers:

$$\left(\begin{array}{c}145144143142141140139138137136135134133\\146101100999897969594939291132\\147102656463626160595857 90131\\148103666373635343332315689130\\149104673817161514133 05588129\\150105683918543122954 87128\\151106694019612112853 86127\end{array}\right)$$

Ulam matrix only with primes, then concatenated and factorized:

$$Up := \left[\begin{array}{ccccccccccccc}0&0&0&0&0&0&139&0&137&0&0&0&0\\0&101&0&0&0&97&0&0&0&0&0&0&0\\0&0&0&0&0&0&61&0&59&0&0&0&131\\0&103&0&37&0&0&0&0&0&31&0&89&0\\149&0&67&0&17&0&0&0&13&0&0&0&0\\0&0&0&0&0&5&0&3&0&29&0&0&0\\151&0&0&0&19&0&0&2&11&0&53&0&127\\0&107&0&41&0&7&0&0&0&0&0&0&0\\0&0&71&0&0&0&23&0&0&0&0&0&0\\0&109&0&43&0&0&0&47&0&0&0&83&0\\0&0&73&0&0&0&0&0&79&0&0&0&0\\0&0&0&113&0&0&0&0&0&0&0&0&0\\157&0&0&0&0&0&163&0&0&0&167&0&0\end{array}\right]$$

$$concM(Up) \to$$

$$\left(\begin{array}{c}1390137\\10100097\\61059000131\\1030370000031089\\14906701700013\\503029\\15100019002110530127\\10704107\\7100023\\1090430004700083\\730000079\\113\\15700000163000167\end{array}\right) factor \to \left(\begin{array}{c}3\cdot 7\cdot 53\cdot 1249\\3^2\cdot 7\cdot 160319\\\boxed{61059000131}\\53\cdot 1117153\cdot 17402221\\3\cdot 107\cdot 46438323053\\41\cdot 12269\\3\cdot 1021\cdot 3203\cdot 1539123815843\\467\cdot 22921\\7\cdot 31\cdot 32719\\3\cdot 83\cdot 599\cdot 7310913133\\\boxed{730000079}\\\boxed{113}\\7\cdot 17\cdot 29\cdot 43\cdot 163\cdot 19949\cdot 32537\end{array}\right)$$

Chapter 8

Conjectures

1. Coloration conjecture: Anyhow all points of an m–dimensional Euclidian space are colored with a finite number of colors, there exists a color which fulfills all distances.

2. Primes: Let a_1, a_2, ..., a_n, be distinct digits, $1 \leq n \leq 9$. How many primes can we construct from all these digits only (eventually repeated)?

3. More generally: when a_1, a_2, ..., a_n, and n are positive integers. Conjecture: Infinitely many!

4. Back concatenated prime sequence: 2, 32, 532, 7532, 117532, 13117532, 1713117532, 191713117532, 23191713117532, ... Conjecture: There are infinitely many primes among the first sequence numbers!

5. Back concatenated odd sequence: 1, 31, 531, 7531, 97531, 1197531, 131197531, 15131197531, 1715131197531, ... Conjecture: There are infinitely many primes among these numbers!

6. Back concatenated even sequence: 2, 42, 642, 8642, 108642, 12108642, 1412108642, 161412108642, ... Conjecture: None of them is a perfect power!

7. Wrong numbers: A number $n = \overline{a_1 a_2 \ldots a_k}$, of at least two digits, with the property: the sequence a_1, a_2, ..., a_k, b_{k+1}, b_{k+2}, ... (where b_{k+i} is the product of the previous k terms, for any $i \geq 1$) contains n as its term.) The authors conjectured that there is no wrong number (!) Therefore, this sequence is empty.

8. Even Sequence is generated by choosing $G = \{2, 4, 6, 8, 10, 12, \ldots\}$, and it is: 2, 24, 246, 2468, 246810, 24681012, ... Searching the first 200 terms of the

sequence we didn't find any n-th perfect power among them, no perfect square, nor even of the form $2p$, where p is a prime or pseudo–prime. Conjecture: There is no n-th perfect power term!

9. Prime-digital sub-sequence "Personal Computer World" Numbers Count of February 1997 presented some of the Smarandache Sequences and related open problems. One of them defines the prime–digital sub–sequence as the ordered set of primes whose digits are all primes: 2, 3, 5, 7, 23, 37, 53, 73, 223, 227, 233, 257, 277, ... We used a computer program in Ubasic to calculate the first 100 terms of the sequence. The 100-th term is 33223. Smith [1996] conjectured that the sequence is infinite. In this paper we will prove that this sequence is in fact infinite.

10. Concatenated Fibonacci sequence: 1, 11, 112, 1123, 11235, 112358, 11235813, 1123581321, 112358132134, ...

11. Back concatenated Fibonacci sequence: 1, 11, 211, 3211, 53211, 853211, 13853211, 2113853211, 342113853211, ... Does any of these numbers is a Fibonacci number? [Marimutha, 1997]

12. Special expressions.

    (a) Perfect powers in special expressions $x^y + y^x$, where $gcd(x, y) = 1$, [Castini, 1995/6, Castillo, 1996/7]. For $x = 1, 2, \ldots, 20$ and $y = 1, 2, \ldots, 20$ one obtains 127 of numbers and following numbers are primes: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 593, 32993, 2097593, 59604644783353249. Kashihara [1996] announced that there are only finitely many numbers of the above form which are products of factorials. In this note we propose the following conjecture: Let a, b, and c three integers with $a \cdot b$ nonzero. Then the equation:

    $$a \cdot x^y + b \cdot y^x = c \cdot z^n ,$$

    with x, y, $n \geq 2$, and $gcd(x, y) = 1$, has finitely many solutions (x, y, z, n). And we prove some particular cases of it, Luca [1997a,b].

    (b) Products of factorials in special expressions. Castillo [1996/7] asked how many primes are there in the n-expression

    $$x_1^{x_2} + x_2^{x_3} + \ldots + x_n^{x_1} , \tag{8.1}$$

    where $n, x_1, x_2, \ldots, x_n > 1$, and $gcd(x_1, x_2, \ldots, x_n) = 1$? For $n = 3$ expression $x_1^{x_2} + x_2^{x_3} + x_3^{x_1}$ has 51 prime numbers: 3, 5, 7, 11, 13, 19, 31, 61, 67, 71, 89, 103, 181, 347, 401, 673, 733, 773,

1301, 2089, 2557, 12497, 33049, 46663, 78857, 98057, 98929, 135329, 262151, 268921, 338323, 390721, 531989, 552241, 794881, 1954097, 2165089, 2985991, 4782977, 5967161, 9765757, 17200609, 35835953, 40356523, 48829699, 387420499, 430513649, 2212731793, 1000000060777, 1000318307057, 1008646564753, where $x_1, x_2, x_3 \in \{1, 2, \ldots, 12\}$. These results were obtained with the following programs:

Program 8.1. of finding the numbers of the form (8.1) for $n = 3$.

$$
P3(a_x, b_x, a_y, b_y, a_z, b_z) := \left|
\begin{array}{l}
j \leftarrow 1 \\
for\ x \in a_x..b_x \\
\quad for\ y \in a_y..b_y \\
\quad\quad for\ z \in a_z..b_z \\
\quad\quad\quad if\ gcd(x, y, z) = 1 \\
\quad\quad\quad\quad \left| \begin{array}{l} se_j \leftarrow x^y + y^z + z^x \\ j \leftarrow j + 1 \end{array} \right. \\
sse \leftarrow sort(se) \\
k \leftarrow 1 \\
s_k \leftarrow sse_1 \\
for\ j \in 2..last(sse) \\
\quad if\ s_k \neq sse_j \\
\quad\quad \left| \begin{array}{l} k \leftarrow k + 1 \\ s_k \leftarrow sse_j \end{array} \right. \\
return\ s
\end{array}
\right.
$$

The program uses Mathcad function *gcd*, greatest common divisor.

Program 8.2. of extraction the prime numbers from a sequences.

$$
IP(s) := \left|
\begin{array}{l}
j \leftarrow 0 \\
for\ k \in 1..last(s) \\
\quad if\ IsPrime(s_k) = 1 \\
\quad\quad \left| \begin{array}{l} j \leftarrow j + 1 \\ ps_j \leftarrow s_k \end{array} \right. \\
return\ ps
\end{array}
\right.
$$

The program uses Mathcad function *IsPrime*.

For $n = 4$ expression $x_1^{x_2} + x_2^{x_3} + x_3^{x_4} + x_4^{x_1}$ has 50 primes: 5, 7, 11, 13, 23, 29, 37, 43, 47, 71, 89, 103, 107, 109, 113, 137, 149, 157, 193, 199, 211, 257, 271, 277, 293, 313, 631, 677, 929, 1031, 1069, 1153, 1321, 1433, 2017, 2161, 3163, 4057, 4337, 4649, 4789, 5399, 6337, 16111, 18757, 28793, 46727, 54521,

64601, 93319, where $x_1, x_2, x_3, x_4 \in \{1, 2, \ldots, 5\}$. These results have been obtained with a program similar to 8.1.

13. There are infinitely many primes which are generalized Smarandache palindromic number GSP1 or GSP2.

Chapter 9

Algorithms

9.1 Constructive Set

9.1.1 Constructive Set of Digits 1 and 2

Definition 9.1.

1. 1, 2 belong to S;
2. if a, b belong to S, then ab belongs to S too;
3. only elements obtained by rules 1. and 2. applied a finite number of times belong to S.

Numbers formed by digits 1 and 2 only: 1, 2, 11, 12, 21, 22, 111, 112, 121, 122, 211, 212, 221, 222, 1111, 1112, 1121, 1122, 1211, 1212, 1221, 1222, 2111, 2112, 2121, 2122, 2211, 2212, 2221, 2222, … .

Remark 9.2.

1. there are 2^k numbers of k digits in the sequence, for $k = 1, 2, 3, \ldots$;
2. to obtain from the k–digits number group the $(k+1)$–digits number group, just put first the digit 1 and second the digit 2 in the front of all k–digits numbers.

9.1.2 Constructive Set of Digits 1, 2 and 3

Definition 9.3.

1. 1, 2, 3 belong to S;

2. if a, b belong to S, then ab belongs to S too;

3. only elements obtained by rules 1. and 2. applied a finite number of times belong to S.

Numbers formed by digits 1, 2, and 3 only: 1, 2, 3, 11, 12, 13, 21, 22, 23, 31, 32, 33, 111, 112, 113, 121, 122, 123, 131, 132, 133, 211, 212, 213, 221, 222, 223, 231, 232, 233, 311, 312, 313, 321, 322, 323, 331, 332, 333, ...

Remark 9.4.

1. there are 3^k numbers of k digits in the sequence, for $k = 1, 2, 3, \ldots$;

2. to obtain from the k–digits number group the $(k+1)$–digits number group, just put first the digit 1, second the digit 2, and third the digit 3 in the front of all k–digits numbers.

9.1.3 Generalized Constructive Set

Program 9.5. for generating the numbers between limits α and β that have the digits from the vector w.

$$Cset(\alpha,\beta,w) := \left| \begin{array}{l} b \leftarrow last(w) \\ j \leftarrow 1 \\ for\ n \in \alpha..\beta \\ \quad \left| \begin{array}{l} d \leftarrow dn(n,b) \\ for\ k \in 1..last(d) \\ \quad wd_k \leftarrow w_{(d_k+1)} \\ cs_j \leftarrow wd \cdot Vb(10,k) \\ j \leftarrow j+1 \end{array} \right. \\ return\ \ cs \end{array} \right.$$

The program uses the subprograms dn, 2.2, and function $Vb(b,m)$ that returns the vector $(b^m\ b^{m-1}\ \ldots b^0)^{\mathrm{T}}$.

1. The first 26 numbers from 0 to 25, with digits 3 to 7 are: 3, 7, 73, 77, 733, 737, 773, 777, 7333, 7337, 7373, 7377, 7733, 7737, 7773, 7777, 73333, 73337, 73373, 73377, 73733, 73737, 73773, 73777, 77333, 77337.

2. The numbers from 0 to 30, with digits 1, 3 and 7 are: 1, 3, 7, 31, 33, 37, 71, 73, 77, 311, 313, 317, 331, 333, 337, 371, 373, 377, 711, 713, 717, 731, 733, 737, 771, 773, 777, 3111, 3113, 3117, 3131.

3. The numbers from 3 to 70, with digits 1, 3, 7 and 9 are: 9, 31, 33, 37, 39, 71, 73, 77, 79, 91, 93, 97, 99, 311, 313, 317, 319, 331, 333, 337, 339, 371, 373, 377, 379, 391, 393, 397, 399, 711, 713, 717, 719, 731, 733, 737, 739, 771, 773, 777, 779, 791, 793, 797, 799, 911, 913, 917, 919, 931, 933, 937, 939, 971, 973, 977, 979, 991, 993, 997, 999, 3111, 3113, 3117, 3119, 3131, 3133, 3137.

4. The numbers from 227 to 280, with digits 1, 2, 3, 7 and 9 are: 2913, 2917, 2919, 2921, 2922, 2923, 2927, 2929, 2931, 2932, 2933, 2937, 2939, 2971, 2972, 2973, 2977, 2979, 2991, 2992, 2993, 2997, 2999, 3111, 3112, 3113, 3117, 3119, 3121, 3122, 3123, 3127, 3129, 3131, 3132, 3133, 3137, 3139, 3171, 3172, 3173, 3177, 3179, 3191, 3192, 3193, 3197, 3199, 3211, 3212, 3213, 3217, 3219, 3221.

9.2 Romanian Multiplication

Another algorithm to multiply two integer numbers, a and b:

- let k be an integer ≥ 2;
- write a and b on two different vertical columns: $col(a)$, respectively $col(b)$;
- multiply a by k, and write the product a_1 on the column $col(a)$;
- divide b by k, and write the integer part of the quotient b_1 on the column $col(b)$;
- ... and so on with the new numbers a_1 and b_1, until we get a $b_i < k$ on the column $col(b)$;

  Then:

- write another column *col*(r), on the right side of *col*(b), such that: for each number of column *col*(b), which may be a multiple of k plus the rest r (where $r \in \{0, 1, 2, \ldots, k-1\}$), the corresponding number on *col*(r) will be r;
- multiply each number of column a by its corresponding r of $col(r)$, and put the new products on another column *col*(p) on the right side of *col*(r);
- finally add all numbers of column *col*(p), $a \times b =$ the sum of all numbers of *col*(p).

Remark 9.6. that any multiplication of integer numbers can be done only by multiplication with $2, 3, \ldots, k$, divisions by k, and additions.

Remark 9.7. This is a generalization of Russian multiplication (when $k = 2$); we call it *Romanian Multiplication*.

This special multiplication is useful when k is very small, the best values being for $k = 2$ (Russian multiplication – known since Egyptian time), or $k = 3$. If k is greater than or equal to $\min\{10, b\}$, this multiplication is trivial (the obvious multiplication).

Program 9.8. for Romanian Multiplication.

$$
RM(a,b,k) := \left|
\begin{array}{l}
w \leftarrow (a\ \ b\ \ \text{"="}\ \ 0) \\
r \leftarrow \ \operatorname{mod}(b,k) \\
Q \leftarrow (a\ \ b\ \ r\ \ a\cdot r) \\
while\ \ b > 1 \\
\quad \left| \begin{array}{l}
a \leftarrow a\cdot k \\
b \leftarrow floor\left(\dfrac{b}{k}\right) \\
r \leftarrow \ \operatorname{mod}(b,k) \\
Q \leftarrow stack[Q, (a\ \ b\ \ r\ \ a\cdot r)]
\end{array}\right. \\
w_{1,4} \leftarrow \sum Q^{\langle 4\rangle} \\
return\ stack(Q, w)
\end{array}\right.
$$

$$
RM(73,97,2) = \begin{pmatrix}
73 & 97 & 1 & 73 \\
146 & 48 & 0 & 0 \\
292 & 24 & 0 & 0 \\
584 & 12 & 0 & 0 \\
1168 & 6 & 0 & 0 \\
2336 & 3 & 1 & 2336 \\
4672 & 1 & 1 & 4672 \\
\hline
73 & \times 97 & \text{"="} & 7081
\end{pmatrix},
$$

$$
RM(73,97,3) = \begin{pmatrix}
73 & 97 & 1 & 73 \\
219 & 32 & 2 & 438 \\
657 & 10 & 1 & 657 \\
1971 & 3 & 0 & 0 \\
5913 & 1 & 1 & 5913 \\
\hline
73 & \times 97 & \text{"="} & 7081
\end{pmatrix},
$$

$$\vdots$$

$$RM(73,97,10) = \left(\begin{array}{rrrr} 73 & 97 & 7 & 511 \\ 730 & 9 & 9 & 6570 \\ 7300 & 0 & 0 & 0 \\ \hline 73 & \times 97 & "=" & 7081 \end{array}\right).$$

$RM(2346789,345793,10) =$

$$\left(\begin{array}{rrrr} 2346789 & 345793 & 3 & 7040367 \\ 23467890 & 34579 & 9 & 211211010 \\ 234678900 & 3457 & 7 & 1642752300 \\ 2346789000 & 345 & 5 & 11733945000 \\ 23467890000 & 34 & 4 & 93871560000 \\ 234678900000 & 3 & 3 & 704036700000 \\ 2346789000000 & 0 & 0 & 0 \\ \hline 2346789 & \times 345793 & "=" & 811503208677 \end{array}\right)$$

Remark 9.9. that any multiplication of integer numbers can be done only by multiplication with 2, 3, ..., 9, 10, divisions by 10, and additions – hence we obtain just the obvious multiplication!

Program 9.10. is the variant that displays the intermediate values of the multiplication process.

$$rm(a,b,k) := \left| \begin{array}{l} s \leftarrow a \cdot \mod(b,k) \\ while\ \ b > 1 \\ \quad \left| \begin{array}{l} a \leftarrow a \cdot k \\ b \leftarrow floor\left(\dfrac{b}{k}\right) \\ s \leftarrow s + a \cdot \mod(b,k) \end{array} \right. \\ return\ \ s \end{array} \right.$$

Example of calling: $rm(2346789,345793,10) = 811503208677$.

Remark 9.11. that any multiplication of integer numbers can be done only by multiplication with 2, 3, ..., 9, 10, divisions by 10, and additions – hence we obtain just the obvious multiplication!

9.3 Division with k to the Power n

Another algorithm to divide an integer number a by k^n, where k, n are integers $ge2$, Bouvier and Michel [1979]:

- write a and k^n on two different vertical columns: $col(a)$, respectively $col(k^n)$;
- divide a by k, and write the integer quotient a_1 on the column $col(a)$;
- divide k^n by k, and write the quotient $q_1 = k^{n-1}$ on the column $col(k^n)$;
- ... and so on with the new numbers a_1 and q_1, until we get $q_n = 1$ $(= k^0)$ on the column $col(k^n)$;

  Then:

- write another column $col(r)$, on the left side of $col(a)$, such that: for each number of column $col(a)$, which may be a multiple of k plus the rest r (where $r \in \{0, 1, 2, ..., k-1\}$), the corresponding number on $col(r)$ will be r;
- write another column $col(p)$, on the left side of $col(r)$, in the following way: the element on line i (except the last line which is 0) will be k^{n-1};
- multiply each number of column $col(p)$ by its corresponding r of $col(r)$, and put the new products on another column $col(r)$ on the left side of $col(p)$;
- finally add all numbers of column $col(r)$ to get the final rest r^n, while the final quotient will be stated in front of $col(k^n)$'s 1. Therefore:

$$\frac{a}{k^n} = a_n \text{ and rest } r_n\,.$$

Remark 9.12. that any division of an integer number by k^n can be done only by divisions to k, calculations of powers of k, multiplications with $1, 2, \ldots, k-1$, additions.

Program 9.13. for division calculation of a positive integer number k of power n where k, n are integers ≥ 2.

$$Dkn(a,k,n) := \left| \begin{array}{l} c_{1,1} \leftarrow a \\ for\ j \in 1..n \\ \quad \left| \begin{array}{l} c_{j,2} \leftarrow \ \mathrm{mod}\ (c_{j,1}, k) \\ c_{j,3} \leftarrow k^{j-1} \cdot c_{j,2} \\ c_{j+1,1} \leftarrow floor\left(\dfrac{c_{j,1}}{k}\right) \end{array} \right. \\ c_{j+1,2} \leftarrow \text{"}rest\text{"} \\ c_{j+1,3} \leftarrow \sum c^{\langle 3 \rangle} \\ return\ c \end{array} \right.$$

The program call *Dkn*, 9.13, for dividing 13537 to 2^7:

$$
Dkn(13537,2,7) = \left(\begin{array}{rcr}
13537 & 1 & 1 \\
6768 & 0 & 0 \\
3384 & 0 & 0 \\
1692 & 0 & 0 \\
846 & 0 & 0 \\
423 & 1 & 32 \\
211 & 1 & 64 \\
\hline
105 & "rest" & 97
\end{array}\right)
$$

The program call *Dkn*, 9.13, for dividing 21345678901 to 3^9:

$$
Dkn(21345678901,3,9) = \left(\begin{array}{rcr}
21345678901 & 1 & 1 \\
7115226300 & 0 & 0 \\
2371742100 & 0 & 0 \\
790580700 & 0 & 0 \\
263526900 & 0 & 0 \\
87842300 & 2 & 486 \\
29280766 & 1 & 729 \\
9760255 & 1 & 2187 \\
3253418 & 2 & 13122 \\
\hline
1084472 & "rest" & 16525
\end{array}\right)
$$

The program call *Dkn*, 9.13, using symbolic computation, for dividing 2536475893647585682919172 to 11^{13}:

$$
Dkn(2536475893647585682919172,11,13) \to
$$

$$
\left(\begin{array}{rcr}
2536475893647585682919172 & 2 & 2 \\
230588717604325971174470 & 10 & 110 \\
20962610691302361015860 & 2 & 242 \\
1905691881027487365078 & 2 & 2662 \\
173244716457044305916 & 7 & 102487 \\
15749519677913118719 & 6 & 966306 \\
1431774516173919883 & 9 & 15944049 \\
130161319652174534 & 9 & 175384539 \\
11832847241106775 & 4 & 857435524 \\
1075713385555161 & 1 & 2357947691 \\
97792125959560 & 10 & 259374246010 \\
8890193269050 & 5 & 1426558353055 \\
808199388095 & 0 & 0 \\
\hline
73472671645 & "rest" & 1689340382677
\end{array}\right)
$$

Program 9.14. for dividing an integer with k^n, where $k, n \in \mathbb{N}^*$, $k, n \geq 2$, without displaying intermediate results of the division.

$$dkn(a,k,n) := \left| \begin{array}{l} R \leftarrow 0 \\ for\ j \in 1..n \\ \quad \left| \begin{array}{l} r \leftarrow \mod(a,k) \\ R \leftarrow R + k^{j-1} \cdot r \\ a \leftarrow floor\left(\frac{a}{k}\right) \end{array} \right. \\ return\ (a\ \text{"}rest\text{"}\ R) \end{array} \right.$$

Examples of dialing the program *dkn*, 9.14:

$$dkn(13537, 2, 7) = (105\ \text{"}rest\text{"}\ 97)\ ,$$

$$dkn(253647589364758568291917 2, 11, 13) \rightarrow (73472671645\ \text{"}rest\text{"}\ 1689340382677)$$

9.4 Generalized Period

Let M be a number in a base b. All distinct digits of M are named generalized period of M. For example, if $M = 104001144$, its generalized period is $g(M) = \{0, 1, 4\}$. Of course, $g(M)$ is included in $\{0, 1, 2, \ldots, b-1\}$.

The number of generalized periods of M is equal to the number of the groups of M such that each group contains all distinct digits of M. For example, $n_g(M) = 2$ because

$$M = \underbrace{104}_{1}\underbrace{001144}_{2}\ .$$

Length of generalized period is equal to the number of its distinct digits. For example, $l_g(M) = 3$.

Questions:

1. Find n_g, l_g for p_n , $n!$, n^n, $\sqrt[n]{n}$.

2. For a given $k \geq 1$, is there an infinite number of primes p_n, or $n!$, or n^n, or $\sqrt[n]{n}$ which have a generalized period of length k? Same question such that the number of generalized periods be equal to k.

3. Let a_1, a_2 , ..., a_h be distinct digits. Is there an infinite number of primes p_n, or $n!$, or n^n, or $\sqrt[n]{n}$ which have as a generalized period the set $\{a_1, a_2, \ldots, a_h\}$?

9.5 Prime Equation Conjecture

Let $k>0$ be an integer. There is only a finite number of solutions in integers p, q, x, y, each greater than 1, to the equation

$$x^p - y^q = k\,. \tag{9.1}$$

For k = 1 this was conjectured by Casseles [1953] and proved by Tijdeman [1976], [Smarandache, 1993a, Ibstedt, 1997].

Lemma 9.15. *Let $,q \geq 2$ be integers and suppose that x, y are nonzero integer that are a solution to equation $x^p - y^q = 1$. Then p and q are necessarily distinct.*

Cassells' theorem is concerned with Catalan's equation for the odd prime exponents p and q. We first prove the easy part of this result.

Proposition 9.16. *Let $p>q$ be two odd primes and suppose that x, y are nonzero integers for which $x^p - y^q = 1$. Then both of the following hold:*

1. $q \mid x$;
2. $|x| \geq q + q^{p-1}$.

Program 9.17. for determining all solutions of the equation (9.1) for p and q give and $k \in \{a_k, a_k+1, \ldots, b_k\}$, $y \in \left\{a_y, a_y+1, \ldots, b_y\right\}$.

$$
Pec(p, a_y, b_y, q, a_k, b_k) := \left|
\begin{array}{l}
return\ \text{"Error."}\ \ if\ \ p \leq q \lor a_y \leq b_y \lor a_k \leq b_k \\
S \leftarrow (\text{"}x\text{"}\ \ \text{"}p\text{"}\ \ \text{"}y\text{"}\ \ \text{"}q\text{"}\ \ \text{"}k\text{"}) \\
for\ k \in a_k..b_k \\
\quad for\ y \in a_y..b_y \\
\quad\quad \left| \begin{array}{l}
xr \leftarrow \sqrt[p]{k+y^q} \\
for\ floor(xr)..ceil(xr) \\
\quad S \leftarrow stack[S, (x\ \ p\ \ y\ \ q\ \ k)]\ \ if\ x^p - y^q = k
\end{array} \right. \\
return\ \ S
\end{array}
\right.
$$

Calling the program *Pec* by command $Pec(5,2,10^3,3,19,2311) =$:

Table 9.1: The solutions of the equation (9.1)

"x"	"p"	"y"	"q"	"k"
2	5	2	3	24
4	5	10	3	24

Continued on next page

"x"	"p"	"y"	"q"	"k"
3	5	6	3	27
3	5	5	3	118
3	5	4	3	179
3	5	3	3	216
3	5	2	3	235
4	5	9	3	295
5	5	14	3	381
4	5	8	3	512
4	5	7	3	681
4	5	6	3	808
4	5	5	3	899
6	5	19	3	917
5	5	13	3	928
4	5	4	3	960
4	5	3	3	997
4	5	2	3	1016
7	5	25	3	1182
5	5	12	3	1397
23	5	186	3	1487
5	5	11	3	1794
6	5	18	3	1944
5	5	10	3	2125

9.5.1 Generalized Prime Equation Conjecture

Let $m \geq 2$ be a positive integer. The Diophantine equation

$$y = 2 \cdot x_1 \cdot x_2 \cdots x_m + k \ , \tag{9.2}$$

has infinitely many solutions in distinct primes y, x_1, x_2, ..., x_m.

Let us remark that $y \in 2\mathbb{N}^* + 1$ and the unknowns $x_1, x_2, \ldots, x_m$ have a similar role.

Program 9.18. for complete solving the equation (9.2) for $m = 3$.

$$
\begin{aligned}
Pecg3(y,k) := \ & S \leftarrow (\text{"}x1\text{"} \ \ \text{"}x2\text{"} \ \ \text{"}x3\text{"}) \\
& for\ x_1 \in 1..y-k-2 \\
& \quad for\ x_2 \in 1+x_1..y-k-1 \\
& \quad \quad for\ x_3 \in 1+x_2..y-k
\end{aligned}
$$

$$\left|\begin{array}{l} \quad\quad S \leftarrow stack[S,(x_1 \;\; x_2 \;\; x_3)] \;\; if \;\; 2 \cdot x_1 \cdot x_2 \cdot x_3 + k = y \\ return \;\; S \end{array}\right.$$

The program is so designed as to avoid getting trivial solutions (for example $x_1 = 1$, $x_2 = 1$ and $x_3 = (y-k)/2$) and symmetrical solutions (for example for $y = 13$ we would have the solutions $x_1 = 1$, $x_2 = 2$ and $x_3 = 3$ but any permutation between these values would be solutions of the equation $2x_1x_2x_3 + 1 = 13$).

Examples of calling the program *Pecg3*:

$$Pecg3(649,1) = \begin{pmatrix} "x1" & "x2" & "x3" \\ 1 & 2 & 162 \\ 1 & 3 & 108 \\ 1 & 4 & 81 \\ 1 & 6 & 54 \\ 1 & 9 & 36 \\ 1 & 12 & 27 \\ 2 & 3 & 54 \\ 2 & 6 & 27 \\ 2 & 9 & 18 \\ 3 & 4 & 27 \\ 3 & 6 & 18 \\ 3 & 9 & 12 \end{pmatrix}$$

$$Pecg3(649,19) = \begin{pmatrix} "x1" & "x2" & "x3" \\ 1 & 3 & 105 \\ 1 & 5 & 63 \\ 1 & 7 & 45 \\ 1 & 9 & 35 \\ 1 & 15 & 21 \\ 3 & 5 & 21 \\ 3 & 7 & 15 \\ 5 & 7 & 9 \end{pmatrix}$$

For $k = 4$, $k = 5, \ldots$ similar programs can be written to determine all untrite and unsymmetrical solutions.

Chapter 10

Documents Mathcad

Prof. dr. Octavian Cira Almost Primes.xmcd 4/23/2015 1 (1)

Almost Primes

ORIGIN := 1

```
AP1(n,L) := | j ← 1
            | a_j ← n
            | for m ∈ n + 1 .. L
            |   | sw ← 0
            |   | for k ∈ 1 .. j
            |   |   if mod(m, a_k) = 0
            |   |     | sw ← 1
            |   |     | break
            |   | if sw = 0
            |   |   | j ← j + 1
            |   |   | a_j ← m
            | return a
```

```
AP2(n,L) := | j ← 1
            | a_j ← n
            | for m ∈ n + 1 .. L
            |   | sw ← 0
            |   | for k ∈ 1 .. j
            |   |   if gcd(m, a_k) ≠ 1
            |   |     | sw ← 1
            |   |     | break
            |   | if sw = 0
            |   |   | j ← j + 1
            |   |   | a_j ← m
            | return a
```

$AP1(10,130)^T = (10\ \ 11\ \ 12\ \ 13\ \ 14\ \ 15\ \ 16\ \ 17\ \ 18\ \ 19\ \ 21\ \ 23\ \ 25\ \ 27\ \ 29\ \ 31\ \ 35\ \ 37\ \ 41\ \ 43\ \ 47\ \ 49\ \ 53\ \ 59\ \ 61\ \ 67\ \ 71\ \ 73\ \ 79\ \ 83\ \ 89\ \ 97\ \ 101\ \ 103\ \ 107\ \ 109\ \ 113\ \ 127)$

$AP2(10,170)^T = (10\ \ 11\ \ 13\ \ 17\ \ 19\ \ 21\ \ 23\ \ 29\ \ 31\ \ 37\ \ 41\ \ 43\ \ 47\ \ 53\ \ 59\ \ 61\ \ 67\ \ 71\ \ 73\ \ 79\ \ 83\ \ 89\ \ 97\ \ 101\ \ 103\ \ 107\ \ 109\ \ 113\ \ 127\ \ 131\ \ 137\ \ 139\ \ 149\ \ 151\ \ 157\ \ 163\ \ 167)$

Figure 10.1: The document Mathcad Almost Primes

Prof. dr. Octavian Cira   Progressions.xmcd   5/10/2015   1 (7)

Progressions

ORIGIN := 1

$prime := (2\ \ 3\ \ 5\ \ 7\ \ 11\ \ 13\ \ 17\ \ 19\ \ 23\ \ 29\ \ 31\ \ 37\ \ 41\ \ 43\ \ 47\ \ 53\ \ 59\ \ 61\ \ 67\ \ 71\ \ 73\ \ 79\ \ 83\ \ 89\ \ 97)^T$

$a := 3 \quad b := 10 \quad k := 1..18 \quad q_k := a \cdot prime_k + b \quad q^T \rightarrow (16\ \ 19\ \ 25\ \ 31\ \ 43\ \ 49\ \ 61\ \ 67\ \ 79\ \ 97\ \ 103\ \ 121\ \ 133\ \ 139\ \ 151\ \ 169\ \ 187\ \ 193)$

$q_k := a^k + b$

$q^T \rightarrow (13\ \ 19\ \ 37\ \ 91\ \ 253\ \ 739\ \ 2197\ \ 6571\ \ 19693\ \ 59059\ \ 177157\ \ 531451\ \ 1594333\ \ 4782979\ \ 14348917\ \ 43046731\ \ 129140173\ \ 387420499)$

$k := 1..12$

$w_k := k^k + 1 \quad w^T \rightarrow (2\ \ 5\ \ 28\ \ 257\ \ 3126\ \ 46657\ \ 823544\ \ 16777217\ \ 387420490\ \ 10000000001\ \ 285311670612\ \ 8916100448257)$

$w_k := k^k - 1 \quad w^T \rightarrow (0\ \ 3\ \ 26\ \ 255\ \ 3124\ \ 46655\ \ 823542\ \ 16777215\ \ 387420488\ \ 9999999999\ \ 285311670610\ \ 8916100448255)$

```
η := READPRN("C:\Users\Tavi\Documents\Mathcad\Mathcad15\Functii pentru numere intregi\Functii Smarandache\VFS.prn")

TS(n) := | return "Err. n<1 sau n nu e intreg"  if  n < 1 ∨ n ≠ trunc(n)
         | if  n > 4
         |    | return 0  if  η_n ≠ n
         |    | return 1  otherwise
         | otherwise
         |    | return 0  if  n = 1 ∨ n = 4
         |    | return 1  otherwise

kP(p, k) := | return 1  if  p = 1
            | return "Err. p not prime"  if  TS(p) = 0
            | q ← mod(p, k + 1)
            | return p  if  q = 0
            | pk ← 1
            | pk ← 2  if  k = 1
            | j ← 1
            | while  prime_j ≤ p
            |    | pk ← pk·prime_j  if  mod(prime_j, k + 1) = q
            |    | j ← j + 1
            | return pk
```

Figure 10.2: The document Mathcad Progression

Prof. dr. Octavian Cira Circular Sequence.xmcd 11/24/2014 1 (2)

Circular Sequence

ORIGIN := 1

1 factor $\rightarrow$ 1

$$p2 := \begin{pmatrix} 12 \\ 21 \end{pmatrix} \quad p2\ \text{factor} \rightarrow \begin{pmatrix} 2^2\cdot 3 \\ 3\cdot 7 \end{pmatrix} \qquad p3 := \begin{pmatrix} 123 \\ 132 \\ 213 \\ 231 \\ 312 \\ 321 \end{pmatrix} \quad p3\ \text{factor} \rightarrow \begin{pmatrix} 3\cdot 41 \\ 2^2\cdot 3\cdot 11 \\ 3\cdot 71 \\ 3\cdot 7\cdot 11 \\ 2^3\cdot 3\cdot 13 \\ 3\cdot 107 \end{pmatrix}$$

$$p4 := \begin{pmatrix} 1234 \\ 1243 \\ 1324 \\ 1342 \\ 1423 \\ 1432 \\ 2134 \\ 2143 \\ 2314 \\ 2341 \\ 2413 \\ 2431 \\ 3124 \\ 3142 \\ 3214 \\ 3241 \\ 3412 \\ 3421 \\ 4123 \\ 4132 \\ 4213 \\ 4231 \\ 4312 \\ 4321 \end{pmatrix} \quad p4\ \text{factor} \rightarrow \begin{pmatrix} 2\cdot 617 \\ 11\cdot 113 \\ 2^2\cdot 331 \\ 2\cdot 11\cdot 61 \\ 1423 \\ 2^3\cdot 179 \\ 2\cdot 11\cdot 97 \\ 2143 \\ 2\cdot 13\cdot 89 \\ 2341 \\ 19\cdot 127 \\ 11\cdot 13\cdot 17 \\ 2^2\cdot 11\cdot 71 \\ 2\cdot 1571 \\ 2\cdot 1607 \\ 7\cdot 463 \\ 2^2\cdot 853 \\ 11\cdot 311 \\ 7\cdot 19\cdot 31 \\ 2^2\cdot 1033 \\ 11\cdot 383 \\ 4231 \\ 2^3\cdot 7^2\cdot 11 \\ 29\cdot 149 \end{pmatrix}$$

Prime Number (1423)
Prime Number (2143)
Prime Number (2341)
Prime Number (4231)

$$p5 := \begin{pmatrix} 12345 & 21345 & 31245 & 41235 & 51234 \\ 12354 & 21354 & 31254 & 41253 & 51243 \\ 12435 & 21435 & 31425 & 41325 & 51324 \\ 12453 & 21453 & 31452 & 41352 & 51342 \\ 12534 & 21534 & 31524 & 41523 & 51423 \\ 12543 & 21543 & 31542 & 41532 & 51432 \\ 13245 & 23145 & 32145 & 42135 & 52134 \\ 13254 & 23154 & 32154 & 42153 & 52143 \\ 13425 & 23415 & 32415 & 42315 & 52314 \\ 13452 & 23451 & 32451 & 42351 & 52341 \\ 13524 & 23514 & 32514 & 42513 & 52413 \\ 13542 & 23541 & 32541 & 42531 & 52431 \\ 14235 & 24135 & 34125 & 43125 & 53124 \\ 14253 & 24153 & 34152 & 43152 & 53142 \\ 14325 & 24315 & 34215 & 43215 & 53214 \\ 14352 & 24351 & 34251 & 43251 & 53241 \\ 14523 & 24513 & 34512 & 43512 & 53412 \\ 14532 & 24531 & 34521 & 43521 & 53421 \\ 15234 & 25134 & 35124 & 45123 & 54123 \\ 15243 & 25143 & 35142 & 45132 & 54132 \\ 15324 & 25314 & 35214 & 45213 & 54213 \\ 15342 & 25341 & 35241 & 45231 & 54231 \\ 15423 & 25413 & 35412 & 45312 & 54312 \\ 15432 & 25431 & 35421 & 45321 & 54321 \end{pmatrix}$$

Figure 10.3: The document Mathcad Circular Sequence

Prof. dr. Octavian Cira   Erdos-Smarandache numbers.xmcd   6/2/2015   1 (4)

Erdos-Smarandache numbers

ORIGIN := 1

Reference:C:\Users\Tavi\Documents\Mathcad\Mathcad15\Conjecturi\Arithmetical functions.xmcd

```
Fa(m) := | f ← (1  1)
         | return f  if m = 1
         | return ("m="  m  "> ca ultimul p^2")  if m > (prime_last(prime))^2
         | j ← 1
         | k ← 0
         | while m ≥ prime_j
         |   | if mod(m, prime_j) = 0
         |   |   | k ← k + 1
         |   |   | m ← m / prime_j
         |   | otherwise
         |   |   | f ← stack[f, (prime_j  k)]  if k > 0
         |   |   | j ← j + 1
         |   |   | k ← 0
         | f ← stack[f, (prime_j  k)]  if k > 0
         | return submatrix(f, 2, rows(f), 1, 2)
```

```
ES(a,b) := | j ← 0
           | for n ∈ a..b
           |   | m ← max(Fa(n)^⟨1⟩)
           |   | if η_n = m
           |   |   | j ← j + 1
           |   |   | s_j ← n
           | return s
```

```
ES2(a,b,nf) := | j ← 0
               | for n ∈ a..b
               |   | m ← max(Fa(n)^⟨1⟩)
               |   | if η_n = m ∧ rows(Fa(n)) = nf
               |   |   | j ← j + 1
               |   |   | s_j ← n
               | return s
```

```
ES1(a,b) := | j ← 0
            | for n ∈ a..b
            |   | m ← max(Fa(n)^⟨1⟩)
            |   | if η_n = m ∧ η_n ≠ n
            |   |   | j ← j + 1
            |   |   | s_j ← n
            | return s
```

$ES(2,60)^T = (2\ 3\ 5\ 6\ 7\ 10\ 11\ 13\ 14\ 15\ 17\ 19\ 20\ 21\ 22\ 23\ 26\ 28\ 29\ 30\ 31\ 33\ 34\ 35\ 37\ 38\ 39\ 40\ 41\ 42\ 43\ 44\ 46\ 47\ 51\ 52\ 53\ 55\ 56\ 57\ 58\ 59\ 60)$

$ES1(2,60)^T = (6\ 10\ 14\ 15\ 20\ 21\ 22\ 26\ 28\ 30\ 33\ 34\ 35\ 38\ 39\ 40\ 42\ 44\ 46\ 51\ 52\ 55\ 56\ 57\ 58\ 60)$

$ES2(2,60,2)^T = (6\ 10\ 14\ 15\ 20\ 21\ 22\ 26\ 28\ 33\ 34\ 35\ 38\ 39\ 40\ 44\ 46\ 51\ 52\ 55\ 56\ 57\ 58)$

$ES2(2,60,3)^T = (30\ 42\ 60)$

$ES2(200,1000,4)^T = (210\ 330\ 390\ 420\ 462\ 510\ 546\ 570\ 630\ 660\ 690\ 714\ 770\ 780\ 798\ 840\ 858\ 870\ 910\ 924\ 930\ 966\ 990)$

$ES2(2000,5000,5)^T = (2310\ 2730\ 3570\ 3990\ 4290\ 4620\ 4830)$

Figure 10.4: The document Mathcad Erdos–Smarandache numbers

Prof. dr. Octavian Cira   Exponents of Power.xmcd   4/23/2015   1 (1)

Exponents of Power *b*

ORIGIN := 1

```
Exp(b,n) := | a ← 0
            | k ← 1
            | while b^k ≤ n
            |   | a ← k  if  mod(n, b^k) = 0
            |   | k ← k + 1
            | return a
```

$n := 1..60 \quad e2_n := \mathrm{Exp}(2,n) \quad e3_n := \mathrm{Exp}(3,n) \quad e5_n := \mathrm{Exp}(5,n)$

$e2^T = (0\ 1\ 0\ 2\ 0\ 1\ 0\ 3\ 0\ 1\ 0\ 2\ 0\ 1\ 0\ 4\ 0\ 1\ 0\ 2\ 0\ 1\ 0\ 3\ 0\ 1\ 0\ 2\ 0\ 1\ 0\ 5\ 0\ 1\ 0\ 2\ 0\ 1\ 0\ 3\ 0\ 1\ 0\ 2\ 0\ 1\ 0\ 4\ 0\ 1\ 0\ 2\ 0\ 1\ 0\ 3\ 0\ 1\ 0\ 2)$

$e3^T = (0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 2\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 2\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 3\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 2\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 2\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 2\ 0\ 0\ 1\ 0\ 0\ 1)$

$e5^T = (0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 2\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 2\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 1)$

Figure 10.5: The document Mathcad Exponents of Power

Indexes

Index of notations

$\mathbb{N} = \{0, 1, 2, \dots\}$;

$\mathbb{N}^* = \mathbb{N} \setminus \{0\} = \{1, 2, \dots\}$;

$\mathbb{P}_{\geq 2} = \{2, 3, 5, 7, 11, 13, \dots\}$, $\mathbb{P}_{\geq 3} = \{3, 5, 7, 11, 13, \dots\}$;

$I_s = \{1, 2, \dots, s\}$: the set of indexes;

$\mathbb{R}$: the real numbers;

$\pi(x)$: the number of prime numbers up to x;

$[x]$: the integer part of number x;

$\{x\}$: the fractional part of x;

$\sigma_k(n)$: the sum of the powers of order k of the divisors of n;

$\sigma(n)$: the sum of the divisors of n; $\sigma(n) = \sigma_1(n)$;

$s(n)$: the sum of the divisors of n without n; $s(n) = \sigma(n) - n$;

$\lfloor a \rfloor$: the lower integer part of a; the greatest integer, smaller than a;

$\lceil a \rceil$: the upper integer part of a; the smallest integer, greater than a;

$n \mid m$: n divides m;

$n \nmid m$: n does not divide m;

(m, n) : the greatest common divisor of m and n; $(m, n) = gcd(m, n)$;

$[m, n]$: the smallest common multiple of m and n; $[m, n] = lcd(m, n)$;

Mathcad Utility Functions

$augment(M,N)$: concatenates matrices M and N that have the same number of lines;

$ceil(x)$: the upper integer part function;

$cols(M)$: the number of columns of matrix M;

$eigenvals(M)$: the eigenvalues of matrix M;

$eigenvec(M,\lambda)$: the eigenvector of matrix M relative to the eigenvalue λ;

$eigenvecs(M)$: the matrix of the eigenvectors of matrix M;

$n\ factor \rightarrow$: symbolic computation function that factorizes n;

$floor(x)$: the lower integer part function;

$gcd(n_1, n_2, \ldots)$: the function which computes the greatest common divisor of $n_1, n_2, \ldots$;

$last(v)$: the last index of vector v;

$lcm(n_1, n_2, \ldots)$: the function which computes the smallest common multiple of $n_1, n_2, \ldots$;

$max(v)$: the maximum of vector v;

$min(v)$: the minimum of vector v;

$\mod(m,n)$: the rest of the division of m by n;

$ORIGIN$: the variable dedicated to the origin of indexes, 0 being an implicit value;

$rref(M)$: determines the matrix *row-reduced echelon form*;

$reverse(M)$: reverses the order of elements in a vector, or of rows in a matrix M.;

$rows(M)$: the number of lines of matrix M;

$solve$: the function of symbolic solving the equations;

$stack(M,N)$: concatenates matrices M and N that have the same care number of columns;

$submatrix(M, k_r, j_r, k_c, j_c)$: extracts from matrix M, from line k_r to line j_r and from column k_c to column j_c, a submatrix;

$trunc(x)$: the truncation function;

$\sum v$: the function that sums the components of vector v .

Mathcad User Arithmetical Functions

ppi: first inferior difference part function, 2.46;

pps: first superior difference part function, 2.47;

Psp: program for determining the numbers $sum - product$ in base b, 2.32;

P_d : function product of all positive divisors of $n_{(10)}$;

P_{dp} : function product of all positive proper divisor of $n_{(10)}$;

Reverse: function returning the inverse of number $n_{(10)}$ in base b, 1.6;

SAOC: sieve of Atkin Optimized by Cira for generating prime numbers, 1.3;

Sgm: program providing the series of maximal gaps;

SEPC: sieve of Erathostenes, linear version of Prithcard, optimized of Cira for generating prime numbers, 1.1;

sp: function digital product in base b of number $n_{(10)}$, 2.90;

scp: function superior cubic part, 2.54;

sfp: function superior factorial part, 2.59;

spi: function inferior fractional square part, 2.51;

sps: function superior fractional square part, 2.52;

spp: function superior prime part, 2.42;

SS: sieve of Sundaram for generating prime numbers, 1.2;

ssp: function superior square part, 2.49;

Z_1 : function pseudo-Smarandache of the order 1, 2.109;

Z_2 : function pseudo-Smarandache of the order 2, 2.113;

Z_3 : function pseudo-Smarandache of the order 3, 2.120;

Mathcad Engendering Programs

Index of name

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Over 300 sequences and many unsolved problems and conjectures related to them are presented herein. These notions, definitions, unsolved problems, questions, theorems corollaries, formulae, conjectures, examples, mathematical criteria, etc. on integer sequences, numbers, quotients, residues, exponents, sieves, pseudo-primes squares cubes factorials, almost primes, mobile periodicals, functions, tables, prime square factorial bases, generalized factorials, generalized palindromes, so on, have been extracted from the Archives of American Mathematics (University of Texas at Austin) and Arizona State University (Tempe): "The Florentin Smarandache papers" special collections, University of Craiova Library, and Arhivele Statului (Filiala Craiova & Filiala Valcea, Romania).

This book was born from the collaboration of the two authors, which started in 2013. The first common work was the volume "Solving Diophantine Equations", published in 2014. The contribution of the authors can be summarized as follows: Florentin Smarandache came with his extraordinary ability to propose new areas of study in number theory, and Octavian Cira - with his algorithmic thinking and knowledge of Mathcad.

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